Updated to 2017-19 Syllabus

CIE IGCSE ADD. MATHS 0606

SUMMARIZED NOTES ON THE EXTENDED SYLLABUS

TABLE OF CONTENTS

1. Set Language & Notation

- A well-defined collection of objects is called a set and each object is called a member or element of the set
- A set is denoted by a capital letter and is expressed by:
  - Listing its elements, e.g. $V = \{a, e, i, o, u\}$
  - A set builder notation

R	set of real numbers
R^+	set of positive real numbers
N	set of natural numbers
Z	set of integers
Z^+	set of positive integers

  - e.g. $\{x: x$ is a prime number and $x < 30\}$
- For any finite set P, $n(P)$ denotes the number of elements in P
- A null or empty set is denote by { } or ∅
- For any two sets P and Q:
  - $P = Q$ if they have the same elements
  - $P \subseteq Q$ if $x \in P \Longrightarrow x \in Q$
  - $P \cap Q = \{x: x \in P$ and $x \in Q\}$
  - $P \cap Q = \emptyset$ then P and Q are disjoint sets
  - $P \cup Q = \{x: x \in P$ or $x \in Q\}$
- For any set P and universal set ξ
  - $P \subseteq \xi$ and $0 \leq n(P) \leq n(\xi)$
  - $P' = \{x: x \in \xi$ and $x \notin P\}$
  - $P \cap P' = \emptyset$
  - $P \cup P' = \xi$

2. Functions

- **One-to-one functions**: each x value maps to one distinct y value

  e.g. $f(x) = 3x - 1$
- **Many-to-one functions:** there are some $f(x)$ values which are generated by more than one x value

  e.g. $f(x) = x^2 - 2x + 3$

  Domain = x values **Range** = y values
- **Notation:** $f(x)$ can also be written as $f: x \mapsto$
- **To find range:**
  - Complete the square

    $x^2 - 2x + 3 \Rightarrow (x-1)^2 + 2$
  - Work out min/max point

    Minimum point $= (1,2)$

∴ all y values are greater than or equal to 2. $f(x) \geq 2$

- One-to-many functions do not exist
- Domain of $g(x)$ = Range of $g^{-1}(x)$
- **Solving functions:**
  - $f(2)$: substitute $x = 2$ and solve for $f(x)$
  - $fg(x)$: substitute $x = g(x)$
  - $f^{-1}(x)$: let $y = f(x)$ and make x the subject
- **Transformation of graphs:**
  - $f(-x)$: reflection in the y-axis
  - $-f(x)$: reflection in the x-axis
  - $f(x) + a$: translation of a units parallel to y-axis
  - $f(x + a)$: translation of $-a$ units parallel to x-axis
  - $f(ax)$: stretch, scale factor $\frac{1}{a}$ parallel to x-axis
  - $af(x)$: stretch, scale factor a parallel to y-axis
- **Modulus function:**
  - Denoted by $|f(x)|$
  - Modulus of a number is its absolute value
  - Never goes below x-axis
  - Makes negative graph into positive by reflecting negative part into x-axis
- **Solving modulus function:**
  - Sketch graphs and find points of intersection
  - Square the equation and solve quadratic
- **Relationship of a function and its inverse:**
  - The graph of the inverse of a function is the reflection of a graph of the function in y=x

3. Quadratic Functions

- **To sketch** $y = ax^2 + bx + c \quad a \neq 0$
  - **Use the turning point:**

    Express $y = ax^2 + bx + c$ as $y = a(x-h)^2 + k$ by completing the square

    $$x^2 + nx \iff \left(x + \frac{n}{2}\right)^2 - \left(\frac{n}{2}\right)^2$$

    $$a(x+n)^2 + k$$

    Where the vertex is $(-n, k)$

    $a > 0$ – u-shaped ∴ minimum point

    $a < 0$ – n-shaped ∴ maximum point
- **Find the x-intercept:**
  - Factorize or use formula
- **Type of root** by calculating discriminant $b^2 - 4ac$
  - If $b^2 - 4ac = 0$, real and equal roots
  - If $b^2 - 4ac > 0$, real and distinct roots
  - If $b^2 - 4ac < 0$, no real roots

- **Intersections of a line and a curve:** if the simultaneous equations of the line and curve leads to a simultaneous equation then:
  - If $b^2 - 4ac = 0$, line is tangent to the curve
  - If $b^2 - 4ac > 0$, line meets curve in two points
  - If $b^2 - 4ac < 0$, line does not meet curve
- Quadratic inequality:
  - $(x-d)(x-\beta) < 0 \Rightarrow d < x < \beta$
  - $(x-d)(x-\beta) > 0 \Rightarrow x < d$ or $x > \beta$

4. Indices & Surds

- **Definitions:**
  - for $a > 0$ and positive integers p and q

$$a^0 = 1 \qquad a^{-p} = \frac{1}{a^p}$$

$$a^{\frac{1}{p}} = \sqrt[p]{a} \qquad a^{\frac{p}{q}} = \left(\sqrt[p]{a}\right)^q$$

- **Rules:**
  - for $a > 0$, $b > 0$ and rational numbers m and n

$$a^m \times a^n = a^{m+n} \qquad a^n \times b^n = (ab)^n$$

$$\frac{a^m}{a^n} = a^{m-n} \qquad \frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

$$(a^m)^n = a^{mn}$$

5. Factors of Polynomials

- To find unknowns in a given identity
  - Substitute suitable values of x

  OR
  - Equalize the given coefficients of like powers of x

Factor Theorem:

- If $(x-t)$ is a factor of the function $p(x)$ then $p(t) = 0$

Remainder Theorem:

- If a function $f(x)$ is divided by $(x-t)$ then:

$$Remainder = f(t)$$

- The formula for remainder theorem:

$$Dividend = Divisor \times Quotient + Remainder$$

6. Simultaneous Equations

- Simultaneous linear equations can be solved either by substitution or elimination
- Simultaneous linear and non-linear equations are generally solved by substitution as follows:
  - Step 1: obtain an equation in one unknown & solve it
  - Step 2: substitute the results from step 1 into the linear equation to find the other unknown
- The points of intersection of two graphs are given by the solution of their simultaneous equations

7. Logarithmic & Exponential Functions

- Definition
  - for $a > 0$ and $a \neq 1$

$$y = a^x \Leftrightarrow x = \log_a y$$

- For $\log_a y$ to be defined

$$y > 0 \text{ and } a > 0,\ a \neq 1$$

- When the logarithms are defined

$$\log_a 1 = 0 \qquad \log_a b + \log_a c \equiv \log_a bc$$

$$\log_a a = 1 \qquad \log_a b - \log_a c \equiv \log_a \frac{b}{c}$$

$$\log_a b \equiv \frac{\log b}{\log a} \qquad \log_a b^n \equiv n \log_a b$$

- When solving logarithmic equations, check solution with original equation and discard any solutions that causes logarithm to be undefined
- Solution of $a^x = b$ where $a \neq -1,\ 0,\ 1$
- If b can be easily written as a^n, then

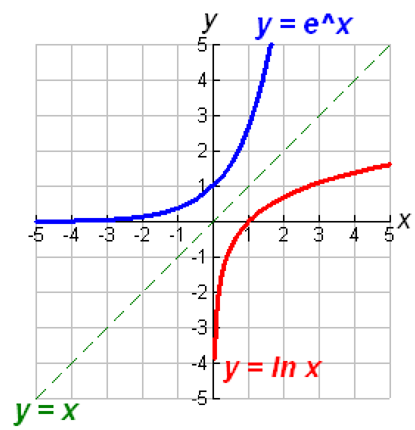
$$a^x = a^n \Rightarrow x = n$$

- Otherwise take logarithms on both sides,

  i.e. $\log a^x = \log b$ and so $x = \frac{\log b}{\log a}$

- $\log \Rightarrow \log_{10}$
- $\ln \Rightarrow \log_e$

Logarithmic & Exponential Graphs



8. Straight Line Graphs

- **Equation of a straight line:**

$$y = mx + c$$

$$y - y_1 = m(x - x_1)$$

- **Gradient:**

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- **Length of a line segment:**

$$\text{Length} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- **Midpoint of a line segment:**

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

- **Parallelogram:**
  - ABCD is a parallelogram ⟺ diagonals AC and BD have a common midpoint
  - Special parallelograms = rhombuses, squares, rectangles
- **Special gradients:**
  - Parallel lines: $m_1 = m_2$
  - Perpendicular lines: $m_1 m_2 = -1$
- **Perpendicular bisector:** line passes through midpoint
- To work out point of intersection of two lines/curves, solve equations simultaneously

9. Circular Measure

- **Radian measure:**

$$\pi = 180^\circ \qquad 2\pi = 360^\circ$$

Degree to Rad $= \times \frac{\pi}{180}$ Rad to Degree $= \times \frac{180}{\pi}$

- **Arc length:**

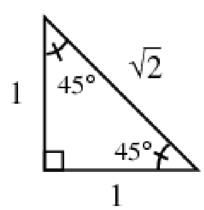
$$s = r\theta$$

- **Area of a sector:**

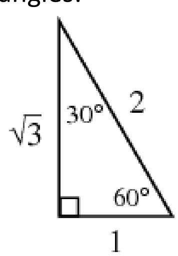
$$A = \frac{1}{2} r^2 \theta$$

10. Trigonometry

- Trigonometric ratio of special angles:

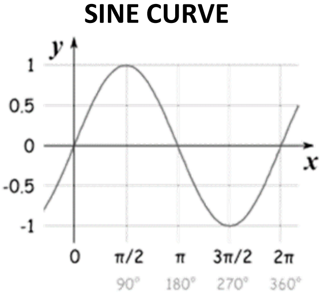


isosceles right triangle

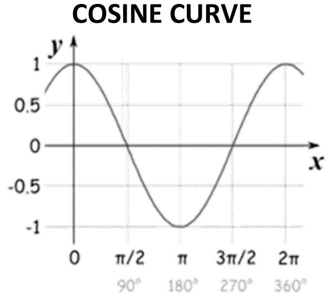


30–60–90° triangle

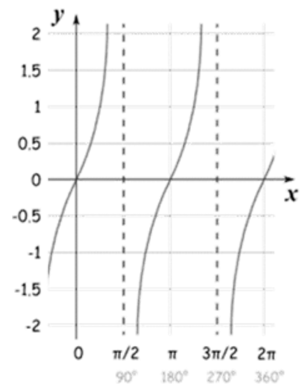
SINE CURVE



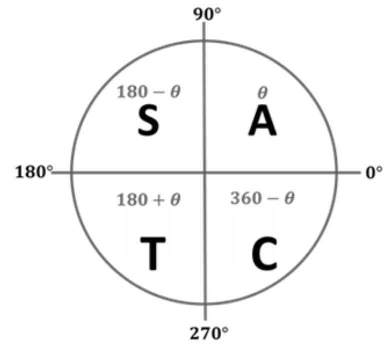
COSINE CURVE



TANGENT CURVE



CAST DIAGRAM



- **Trigonometric ratios:**

$$\sec\theta = \frac{1}{\cos\theta} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

- **Trigonometric identities:**

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sin^2\theta + \cos^2\theta = 1$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta \qquad \tan^2\theta + 1 = \sec^2\theta$$

- **Sketching trigonometric graphs:**

$$y = a(\ldots bx + c) - d$$

changes amplitude

increases no. of cycles

alters x-axis by $-c$

alters y-axis by d

11. Permutations & Combinations

- **Basic Counting Principle:** to find the number of ways of performing several tasks in succession, multiply the number of ways in which each task can be performed:

  e.g. $5 \times 4 \times 3 \times 2$

- **Factorial:** $n! = n \times (n-1) \times (n-2) \ldots \times 3 \times 2 \times 1$
  - NOTE: $0! = 1$
- **Permutations:**
  - The number of ordered arrangements of r objects taken from n unlike objects is:

$$^nP_r = \frac{n!}{(n-r)!}$$

  - Order **matters**
- **Combinations:**
  - The number of ways of selecting r objects from n unlike objects is:

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

- Order **does not** matter

12. Binomial Expansions

- The binomial theorem allows expansion of any expression in the form $(a+b)^n$

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \cdots + {}^nC_ny^n$$

- e.g. Expand $(2x-1)^4$

$$(2x-1)^4 = {}^4C_0(2x)^4 + {}^4C_1(2x)^3(-1)$$
$$+{}^4C_2(2x)^2(-1)^2 + {}^4C_3(2x)\,(-1)^3 + {}^4C_4(-1)^4$$
$$= 1(2x)^4 + 4(2x)^3(-1) + 6(2x)^2(-1)^2 +$$
$$4(2x)\,(-1)^3 + 1(-1)^4$$
$$= 16x^4 - 32x^3 + 24x^2 - 8x + 1$$

- The powers of x are in descending order

13. Vectors in 2 Dimensions

- **Position vector:** position of point relative to origin, $\overrightarrow{OP}$
- **Forms of vector:**

$$\begin{pmatrix} a \\ b \end{pmatrix} \qquad \overrightarrow{AB} \qquad p \qquad a\mathrm{i} - b\mathrm{j}$$

- **Parallel vectors:** same direction but different magnitude
- **Generally,** $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
- **Magnitude** $= \sqrt{\mathrm{i}^2 + \mathrm{j}^2}$
- **Unit vectors:** vectors of magnitude 1
  - Examples: consider vector $\overrightarrow{AB}$

$$\overrightarrow{AB} = 2\mathrm{i} + 3\mathrm{j} \qquad \left|\overrightarrow{AB}\right| = \sqrt{13}$$
$$\therefore Unit\ vector\ = \frac{1}{\sqrt{13}}(2i + 3j)$$

- **Collinear vectors:** vectors on the same line
- **Dot product:**

$$(a\boldsymbol{i} + b\boldsymbol{j}).(c\boldsymbol{i} + d\boldsymbol{j}) = (ac\boldsymbol{i} + bd\boldsymbol{j})$$

- **Angle between two diverging vectors:**

$$\cos A = \frac{a.b}{|a||b|}$$

<u>**Relative Velocity**</u>

- **Motion in the water:**

$$V_w = \ true\ velocity\ of\ water$$
$$V_{P/W}\ = \ velocity\ of\ P\ relative\ to\ W - \ still\ water$$

- Course taken by P is direction of $V_{P/W}$
- **Motion in the air:**

$$V_w = \ true\ velocity\ of\ wind\ or\ air$$
$$V_{P/W}\ = \ velocity\ of\ P\ relative\ to\ W - \ still\ wind/air$$

- Course take by P is direction of $V_{P/W}$

$$V_{P/Q} = V_P - V_Q$$

14. Matrices

- **Order of a matrix:** a matrix with m rows and n columns, Order $= m \times n$
- **Adding/subtracting matrices:** add/subtract each corresponding element
- **Scalar multiplication:** to multiply a matrix by k, multiply each element by k
- **Multiplying matrices:** multiply row by column
- **Identity matric:**

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad IA = A \text{ and } AI = I$$

- **Calculating the determinant:**

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad |A| = (ad - bc)$$

- **Inverse of a 2 by 2 matrix:**
  - Switch leading diagonal, negate secondary diagonal
  - Multiply by $\frac{1}{|A|}$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad A^{-1}A = AA^{-1} = I$$

- Solving simultaneous linear equations by a matrix method:

$$ax + by = h \qquad cx + dy = k$$

- Equation can be written as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} h \\ k \end{pmatrix}$$

- Rearrange it and solve:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}$$

- **For a matrix to give unique solutions:**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$$

15. Differentiation & Integration

<u>*15.1 Differentiation*</u>

FUNCTION	1ST DERIVATIVE	2ND DERIVATIVE
$y = x^n$	$\frac{dy}{dx} = nx^{n-1}$	$\frac{d^2y}{dx^2} = n(n-1)x^{n-2}$

INCREASING FUNCTION	DECREASING FUNCTION
$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$

- **Stationary point:** equate first derivative to zero

$$\frac{dy}{dx} = 0$$

- **2nd Derivative:** finds nature of the stationary point
  - If value +ve, min. point → negative stationary point
  - If value –ve, max. point → positive stationary point
- **Chain rule:**

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

- **Product rule:**

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

- **Quotient rule:**

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Special Differentials

$\frac{dy}{dx}$ *of* $\sin ax = a\cos ax$
$\frac{dy}{dx}$ *of* $\cos ax = -a\sin ax$
$\frac{dy}{dx}$ *of* $\tan ax = a\sec^2 ax$
$\frac{dy}{dx}$ *of* $e^{ax+b} = ae^{ax+b}$
$\frac{dy}{dx}$ *of* $\ln x = \frac{1}{x}$
$\frac{dy}{dx}$ *of* $\ln(f(x)) = \frac{f'(x)}{f(x)}$

- **Related rates of change:**
  - If x and y are related by the equation $y = f(x)$, then the rates of change $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are related by:

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

- **Small changes:**
  - If $y = f(x)$ and small change δx in x causes a small change δy in y, then

$$\delta y \approx \left(\frac{dy}{dx}\right)_{x=k} \times \delta x$$

15.2 Integration

$$\int ax^n = a\frac{x^{n+1}}{(n+1)} + c$$

$$\int (ax+b)^n = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

- **Definite integral:** substitute coordinates/values & find c
- **Integrating by parts:**

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}dx$$

  - What to make u: **LATE**

- **To find area under the graph (curve and x-axis):**
  - Integrate curve
  - Substitute boundaries of x
  - Subtract one from another (ignore c)

$$\int_c^d y\,dx$$

- **To find volume under the graph (curve and x-axis):**
  - Square the function
  - Integrate and substitute
  - Multiply by π

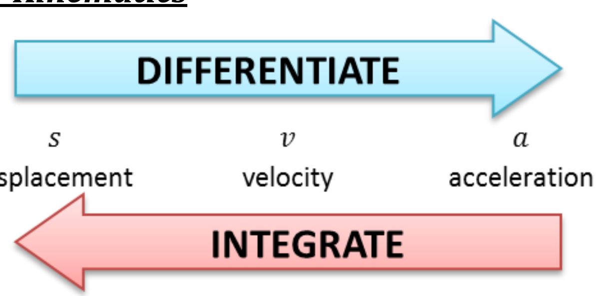
$$\int_c^d \pi y^2\,dx$$

- **To find area/volume between curve and y-axis:**
  - Make x subject of the formula
  - Follow above method using y-values instead of x-values

Special Integrals

$\int \sin(ax+b) = -\frac{1}{a}\cos(ax+b) + c$
$\int \cos(ax+b) = \frac{1}{a}\sin(ax+b) + c$
$\int \sec^2(ax+b) = \frac{1}{a}\tan(ax+b) + c$
$\int \frac{1}{ax+b} = \frac{1}{a}\ln\lvert ax+b\rvert + c$
$\int e^{ax+b} = \frac{1}{a}e^{ax+b} + c$

15.3 Kinematics



- Particle at instantaneous rest, $v = 0$
- Maximum displacement from origin, $v = 0$
- Maximum velocity, $a = 0$

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