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Cambridge
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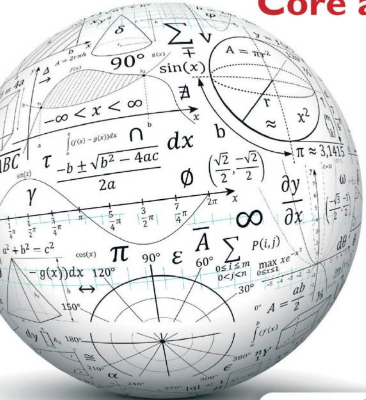
Mathematics
Core and Extended

Third Edition



Ric Pimentel
Terry Wall

COMPANION CD

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Contents

Introduction

To the Student

This textbook has been written by two experienced mathematics teachers.

The book is written to cover every section of the Cambridge IGCSE® Mathematics (0580) syllabus (Core and Extended). The syllabus headings (Number, Algebra and graphs, Geometry, Mensuration, Coordinate geometry, Trigonometry, Matrices and transformations, Probability, Statistics) are mirrored in the textbook. Each major topic is divided into a number of chapters, and each chapter has its own discrete exercises and student assessments. The Core sections are identified with a green band and the Extended with a red band. Students using this book may follow either a Core or Extended curriculum.

The syllabus specifically refers to 'Applying mathematical techniques to solve problems' and this is fully integrated into the exercises and assessments. This book also includes a number of such problems so that students develop their skills in this area throughout the course. Ideas for ICT activities are also included, although this is not part of the examination.

The CD included with this book contains Personal Tutor audio-visual worked examples covering the main concepts.

The study of mathematics crosses all lands and cultures. A mathematician in Africa may be working with another in Japan to extend work done by a Brazilian in the USA; art, music, language and literature belong to the culture of the country of origin. Opera is European. Noh plays are Japanese. It is not likely that people from different cultures could work together on a piece of Indian art for example. But all people in all cultures have tried to understand their world, and mathematics has been a common way of furthering that understanding, even in cultures which have left no written records. Each Topic in this textbook has an introduction which tries to show how, over a period of thousands of years, mathematical ideas have been passed from one culture to another.

The Ishango Bone from Stone-Age Africa has marks suggesting it was a tally stick. It was the start of arithmetic. 4500 years ago in ancient Mesopotamia, clay tablets show multiplication and division problems. An early abacus may have been used at this time. 3600 years ago what is now called The Rhind Papyrus was found in Egypt. It shows simple algebra and fractions. The Moscow Papyrus shows how to find the volume of a pyramid. The Egyptians advanced our knowledge of geometry. The Babylonians worked with arithmetic. 3000 years ago in India the great wise men advanced mathematics and their knowledge travelled to Egypt and later to Greece, then to the rest of Europe when great Arab mathematicians took their knowledge with them to Spain.

Europeans and later Americans made mathematical discoveries from the fifteenth century. It is likely that, with the re-emergence of China and India as major world powers, these countries will again provide great mathematicians and the cycle will be completed. So when you are studying from this textbook remember that you are following in the footsteps of earlier mathematicians who were excited by the discoveries they had made. These discoveries changed our world.

You may find some of the questions in this book difficult. It is easy when this happens to ask the teacher for help. Remember though that mathematics is intended to stretch the mind. If you are trying to get physically fit, you do not stop as soon as things get hard. It is the same with mental fitness. Think logically, try harder. You can solve that difficult problem and get the feeling of satisfaction that comes with learning something new.

Ric Pimentel
Terry Wall

Topic 1

Number

Syllabus

E1.1

Identify and use natural numbers, integers (positive, negative and zero), prime numbers, square numbers, common factors and common multiples, rational and irrational numbers (e.g. π, $\sqrt{2}$), real numbers.

E1.2

Use language, notation and Venn diagrams to describe sets and represent relationships between sets as follows:
Definition of sets
e.g. $A = \{x: x \text{ is a natural number}\}$
$B = \{(x, y): y = mx + c\}$
$C = \{x: a \leq x \leq b\}$
$D = \{a, b, c, \ldots\}$

E1.3

Calculate squares, square roots, cubes and cube roots of numbers.

E1.4

Use directed numbers in practical situations.

E1.5

Use the language and notation of simple vulgar and decimal fractions and percentages in appropriate contexts.
Recognise equivalence and convert between these forms.

E1.6

Order quantities by magnitude and demonstrate familiarity with the symbols $=, \neq, >, <, \geq, \leq$

E1.7

Understand the meaning and rules of indices.
Use the standard form $A \times 10^n$ where n is a positive or negative integer, and $1 \leq A < 10$.

E1.8

Use the four rules for calculations with whole numbers, decimals and vulgar (and mixed) fractions, including correct ordering of operations and use of brackets.

E1.9

Make estimates of numbers, quantities and lengths, give approximations to specified numbers of significant figures and decimal places and round off answers to reasonable accuracy in the context of a given problem.

E1.10

Give appropriate upper and lower bounds for data given to a specified accuracy.
Obtain appropriate upper and lower bounds to solutions of simple problems given data to a specified accuracy.

E1.11

Demonstrate an understanding of ratio and proportion.
Increase and decrease a quantity by a given ratio.
Use common measures of rate.
Calculate average speed.

E1.12

Calculate a given percentage of a quantity.
Express one quantity as a percentage of another.
Calculate percentage increase or decrease.
Carry out calculations involving reverse percentages.

E1.13

Use a calculator efficiently.
Apply appropriate checks of accuracy.

E1.14

Calculate times in terms of the 24-hour and 12-hour clock.
Read clocks, dials and timetables.

E1.15

Calculate using money and convert from one currency to another.

E1.16

Use given data to solve problems on personal and household finance involving earnings, simple interest and compound interest.
Extract data from tables and charts.

E1.17

Use exponential growth and decay in relation to population and finance.

Contents

Hindu mathematicians

In 1300BCE a Hindu teacher named Laghada used geometry and trigonometry for his astronomical calculations. At around this time, other Indian mathematicians solved quadratic and simultaneous equations.

Much later, in about AD500, another Indian teacher, Aryabhata, worked on approximations for π (pi) and on the trigonometry of the sphere. He realised that not only did the planets go round the Sun but that their paths were elliptic.

Brahmagupta, a Hindu, was the first to treat zero as a number in its own right. This helped to develop the decimal system of numbers.

One of the greatest mathematicians of all time was Bhascara who, in the twelfth century, worked in algebra and trigonometry. He discovered that:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

His work was taken to Arabia and later to Europe.

Aryabhata (476–550)

1 Number and language

Vocabulary for sets of numbers

A square can be classified in many different ways. It is a quadrilateral but it is also a polygon and a two-dimensional shape. Just as shapes can be classified in many different ways, so can numbers. Below is a description of some of the more common types of numbers.

Natural numbers

A child learns to count 'one, two, three, four, …'. These are sometimes called the counting numbers or whole numbers.

The child will say 'I am three', or 'I live at number 73'.

If we include the number 0, then we have the set of numbers called the **natural numbers**.

The set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$.

Integers

On a cold day, the temperature may be 4 °C at 10 p.m. If the temperature drops by a further 6 °C, then the temperature is 'below zero'; it is −2 °C.

If you are overdrawn at the bank by $200, this might be shown as −$200.

The set of **integers** $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

$\mathbb{Z}$ is therefore an extension of $\mathbb{N}$. Every natural number is an integer.

Rational numbers

A child may say 'I am three'; she may also say 'I am three and a half', or even 'three and a quarter'. $3\frac{1}{2}$ and $3\frac{1}{4}$ are **rational numbers**. All rational numbers can be written as a fraction whose denominator is not zero. All terminating and recurring decimals are rational numbers as they can also be written as fractions, e.g.

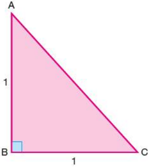
$$0.2 = \frac{1}{5} \quad 0.3 = \frac{3}{10} \quad 7 = \frac{7}{1} \quad 1.53 = \frac{153}{100} \quad 0.\dot{2} = \frac{2}{9}$$

The set of rational numbers $\mathbb{Q}$ is an extension of the set of integers.

Real numbers

Numbers which cannot be expressed as a fraction are not rational numbers; they are **irrational numbers**.

Using Pythagoras' rule in the diagram to the left, the length of the hypotenuse AC is found as:



$$AC^2 = 1^2 + 1^2 = 2$$
$$AC = \sqrt{2}$$

$\sqrt{2} = 1.41421356\ldots$. The digits in this number do not recur or repeat in a pattern. This is a property of all irrational numbers. Another example of an irrational number you will come across is π (pi). It is the ratio of the circumference of a circle to the length of its diameter. Although it is often rounded to 3.142, the digits continue indefinitely never repeating themselves in any particular pattern.

The set of rational and irrational numbers together form the set of **real numbers** $\mathbb{R}$.

Prime numbers

A prime number is one whose only factors are 1 and itself. (Note that 1 is not a prime number.)

Exercise 1.1

1. In a 10 by 10 square, write the numbers 1 to 100.
Cross out number 1.
Cross out all the even numbers after 2 (these have 2 as a factor).
Cross out every third number after 3 (these have 3 as a factor).
Continue with 5, 7, 11 and 13, then list all the prime numbers less than 100.

Square numbers

The number 1 can be written as 1×1 or 1^2.
The number 4 can be written as 2×2 or 2^2.
9 can be written as 3×3 or 3^2.
16 can be written as 4×4 or 4^2.

When an integer (whole number) is multiplied by itself, the result is a square number. In the examples above, 1, 4, 9 and 16 are all square numbers.

Cube numbers

The number 1 can be written as $1 \times 1 \times 1$ or 1^3.
The number 8 can be written as $2 \times 2 \times 2$ or 2^3.
27 can be written as $3 \times 3 \times 3$ or 3^3.
64 can be written as $4 \times 4 \times 4$ or 4^3.

When an integer is multiplied by itself and then by itself again, the result is a cube number. In the examples above 1, 8, 27 and 64 are all cube numbers.

Factors

The factors of 12 are all the numbers which will divide exactly into 12, i.e. 1, 2, 3, 4, 6 and 12.

Exercise 1.2

1. List all the factors of the following numbers:

a) 6	b) 9	c) 7	d) 15	e) 24
f) 36	g) 35	h) 25	i) 42	j) 100

Prime factors

The factors of 12 are 1, 2, 3, 4, 6 and 12.

Of these, 2 and 3 are prime numbers, so 2 and 3 are the prime factors of 12.

Exercise 1.3

1. List the prime factors of the following numbers:

a) 15	b) 18	c) 24	d) 16	e) 20
f) 13	g) 33	h) 35	i) 70	j) 56

An easy way to find prime factors is to divide by the prime numbers in order, smallest first.

Worked examples

a) Find the prime factors of 18 and express it as a product of prime numbers:

	18
2	9
3	3
3	1

$18 = 2 \times 3 \times 3$ or 2×3^2

b) Find the prime factors of 24 and express it as a product of prime numbers:

	24
2	12
2	6
2	3
3	1

$24 = 2 \times 2 \times 2 \times 3$ or $2^3 \times 3$

c) Find the prime factors of 75 and express it as a product of prime numbers:

	75
3	25
5	5
5	1

$75 = 3 \times 5 \times 5$ or 3×5^2

Exercise 1.4

1. Find the prime factors of the following numbers and express them as a product of prime numbers:

a)	12	b)	32	c)	36	d)	40	e)	44
f)	56	g)	45	h)	39	i)	231	j)	63

● Highest common factor

The prime factors of 12 are $2 \times 2 \times 3$.
The prime factors of 18 are $2 \times 3 \times 3$.

So the highest common factor (HCF) can be seen by inspection to be 2×3, i.e. 6.

● Multiples

Multiples of 5 are 5, 10, 15, 20, etc.

The lowest common multiple (LCM) of 2 and 3 is 6, since 6 is the smallest number divisible by 2 and 3.

The LCM of 3 and 5 is 15.

The LCM of 6 and 10 is 30.

Exercise 1.5

1. Find the HCF of the following numbers:

a)	8, 12	b)	10, 25	c)	12, 18, 24
d)	15, 21, 27	e)	36, 63, 108	f)	22, 110
g)	32, 56, 72	h)	39, 52	i)	34, 51, 68
j)	60, 144				

2. Find the LCM of the following:

a)	6, 14	b)	4, 15	c)	2, 7, 10	d)	3, 9, 10
e)	6, 8, 20	f)	3, 5, 7	g)	4, 5, 10	h)	3, 7, 11
i)	6, 10, 16	j)	25, 40, 100				

● Rational and irrational numbers

A **rational number** is any number which can be expressed as a fraction. Examples of some rational numbers and how they can be expressed as a fraction are shown below:

$$0.2 = \frac{1}{5} \quad 0.3 = \frac{3}{10} \quad 7 = \frac{7}{1} \quad 1.53 = \frac{153}{100} \quad 0.\dot{2} = \frac{2}{9}$$

An **irrational number** cannot be expressed as a fraction. Examples of irrational numbers include:

$$\sqrt{2}, \quad \sqrt{5}, \quad 6 - \sqrt{3}, \quad \pi$$

In summary:

Rational numbers include:

- whole numbers,
- fractions,
- recurring decimals,
- terminating decimals.

Irrational numbers include:

- the square root of any number other than square numbers,
- a decimal which neither repeats nor terminates (e.g. π).

Exercise 1.6

1. For each of the numbers shown below state whether it is rational or irrational:

a) 1.3 b) $0.\dot{6}$ c) $\sqrt{3}$

d) $-2\frac{3}{5}$ e) $\sqrt{25}$ f) $\sqrt[3]{8}$

g) $\sqrt{7}$ h) 0.625 i) $0.\dot{1}\dot{1}$

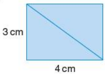
2. For each of the numbers shown below state whether it is rational or irrational:

a) $\sqrt{2} \times \sqrt{3}$ b) $\sqrt{2} + \sqrt{3}$ c) $(\sqrt{2} \times \sqrt{3})^2$

d) $\dfrac{\sqrt{8}}{\sqrt{2}}$ e) $\dfrac{2\sqrt{5}}{2\sqrt{20}}$ f) $4 + (\sqrt{9} - 4)$

3. In each of the following decide whether the quantity required is rational or irrational. Give reasons for your answer.

a)



The length of the diagonal

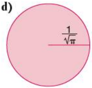
b)

The circumference of the circle

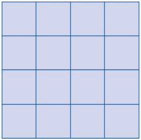
c)

The side length of the square

d)



The area of the circle



Square roots

The square on the left contains 16 squares. It has sides of length 4 units.

So the square root of 16 is 4.

This can be written as $\sqrt{16} = 4$.

Note that 4×4 is 16 so 4 is the square root of 16.

However, -4×-4 is also 16 so -4 is also the square root of 16.

By convention, $\sqrt{16}$ means 'the positive square root of 16' so $\sqrt{16} = 4$ but the square root of 16 is ± 4 i.e. $+4$ or -4.

Note that -16 has no square root since any integer squared is positive.

Exercise 1.7

1. Find the following:

   a) $\sqrt{25}$ b) $\sqrt{9}$ c) $\sqrt{49}$ d) $\sqrt{100}$
   e) $\sqrt{121}$ f) $\sqrt{169}$ g) $\sqrt{0.01}$ h) $\sqrt{0.04}$
   i) $\sqrt{0.09}$ j) $\sqrt{0.25}$

2. Use the $\sqrt{\ }$ key on your calculator to check your answers to question 1.

3. Calculate the following:

   a) $\sqrt{\frac{1}{9}}$ b) $\sqrt{\frac{1}{16}}$ c) $\sqrt{\frac{1}{25}}$ d) $\sqrt{\frac{1}{49}}$
   e) $\sqrt{\frac{1}{100}}$ f) $\sqrt{\frac{4}{9}}$ g) $\sqrt{\frac{9}{100}}$ h) $\sqrt{\frac{49}{81}}$
   i) $\sqrt{2\frac{7}{9}}$ j) $\sqrt{6\frac{1}{4}}$

Exercise 1.8

Using a graph

1. Copy and complete the table below for the equation $y = \sqrt{x}$.

x	0	1	4	9	16	25	36	49	64	81	100
y											

   Plot the graph of $y = \sqrt{x}$. Use your graph to find the approximate values of the following:

   a) $\sqrt{35}$ b) $\sqrt{45}$ c) $\sqrt{55}$ d) $\sqrt{60}$ e) $\sqrt{2}$

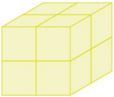
2. Check your answers to question 1 above by using the $\sqrt{\ }$ key on a calculator.

Cube roots

The cube below has sides of 2 units and occupies 8 cubic units of space. (That is, $2 \times 2 \times 2$.)

So the cube root of 8 is 2.

This can be written as $\sqrt[3]{8} = 2$.



$\sqrt[3]{\ }$ is read as 'the cube root of ...'.

$\sqrt[3]{64}$ is 4, since $4 \times 4 \times 4 = 64$.

Note that $\sqrt[3]{64}$ is not -4

since $-4 \times -4 \times -4 = -64$

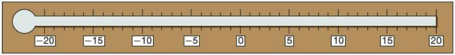
but $\sqrt[3]{-64}$ is -4.

Exercise 1.9

1. Find the following cube roots:

   a) $\sqrt[3]{8}$ b) $\sqrt[3]{125}$ c) $\sqrt[3]{27}$ d) $\sqrt[3]{0.001}$
   e) $\sqrt[3]{0.027}$ f) $\sqrt[3]{216}$ g) $\sqrt[3]{1000}$ h) $\sqrt[3]{1\,000\,000}$
   i) $\sqrt[3]{-8}$ j) $\sqrt[3]{-27}$ k) $\sqrt[3]{-1000}$ l) $\sqrt[3]{-1}$

Directed numbers



Worked example The diagram above shows the scale of a thermometer. The temperature at 0400 was −3 °C. By 0900 the temperature had risen by 8 °C. What was the temperature at 0900?

$(-3)^\circ + (8)^\circ = (5)^\circ$

Exercise 1.10

1. The highest temperature ever recorded was in Libya. It was 58 °C. The lowest temperature ever recorded was −88 °C in Antarctica. What is the temperature difference?

2. My bank account shows a credit balance of $105. Describe my balance as a positive or negative number after each of these transactions is made in sequence:

   a) rent $140 b) car insurance $283
   c) 1 week's salary $230 d) food bill $72
   e) credit transfer $250

3. The roof of an apartment block is 130 m above ground level. The car park beneath the apartment is 35 m below ground level. How high is the roof above the floor of the car park?

4. A submarine is at a depth of 165 m. If the ocean floor is 860 m from the surface, how far is the submarine from the ocean floor?

Student assessment 1

1. State whether the following numbers are rational or irrational:

a) 1.5 b) $\sqrt{7}$ c) $0.\dot{7}$

d) $0.\dot{7}\dot{3}$ e) $\sqrt{121}$ f) π

2. Show, by expressing them as fractions or whole numbers, that the following numbers are rational:

a) 0.625 b) $\sqrt[3]{27}$ c) 0.44

3. Find the value of:

a) 9^2 b) 15^2 c) $(0.2)^2$ d) $(0.7)^2$

4. Calculate:

a) $(3.5)^2$ b) $(4.1)^2$ c) $(0.15)^2$

5. Without using a calculator, find:

a) $\sqrt{225}$ b) $\sqrt{0.01}$ c) $\sqrt{0.81}$

d) $\sqrt{\frac{9}{25}}$ e) $\sqrt{5\frac{4}{9}}$ f) $\sqrt{2\frac{23}{49}}$

6. Without using a calculator, find:

a) 4^3 b) $(0.1)^3$ c) $(\frac{2}{3})^3$

7. Without using a calculator, find:

a) $\sqrt[3]{27}$ b) $\sqrt[3]{1\,000\,000}$ c) $\sqrt[3]{\frac{64}{125}}$

8. My bank statement for seven days in October is shown below:

Date	Payments ($)	Receipts ($)	Balance ($)
01/10			200
02/10	284		(a)
03/10		175	(b)
04/01	(c)		46
05/10		(d)	120
06/10	163		(e)
07/10		28	(f)

Copy and complete the statement by entering the amounts (a) to (f).

2 Accuracy

Approximation

In many instances exact numbers are not necessary or even desirable. In those circumstances approximations are given. The approximations can take several forms. The common types of approximation are dealt with below.

Rounding

If 28 617 people attend a gymnastics competition, this figure can be reported to various levels of accuracy.

To the nearest 10 000 this figure would be rounded up to 30 000.
To the nearest 1000 the figure would be rounded up to 29 000.
To the nearest 100 the figure would be rounded down to 28 600.

In this type of situation it is unlikely that the exact number would be reported.

Exercise 2.1

1. Round the following numbers to the nearest 1000:

a) 68 786	b) 74 245	c) 89 000
d) 4020	e) 99 500	f) 999 999

2. Round the following numbers to the nearest 100:

a) 78 540	b) 6858	c) 14 099
d) 8084	e) 950	f) 2984

3. Round the following numbers to the nearest 10:

a) 485	b) 692	c) 8847
d) 83	e) 4	f) 997

Decimal places

A number can also be approximated to a given number of decimal places (d.p.). This refers to the number of digits written after a decimal point.

Worked examples

a) Write 7.864 to 1 d.p.
The answer needs to be written with one digit after the decimal point. However, to do this, the second digit after the decimal point also needs to be considered. If it is 5 or more then the first digit is rounded up.

i.e. 7.864 is written as 7.9 to 1 d.p.

b) Write 5.574 to 2 d.p.
The answer here is to be given with two digits after the decimal point. In this case the third digit after the decimal point needs to be considered. As the third digit after the decimal point is less than 5, the second digit is not rounded up.

i.e. 5.574 is written as 5.57 to 2 d.p.

Exercise 2.2

1. Give the following to 1 d.p.

a) 5.58	b) 0.73	c) 11.86
d) 157.39	e) 4.04	f) 15.045
g) 2.95	h) 0.98	i) 12.049

2. Give the following to 2 d.p.

a) 6.473	b) 9.587	c) 16.476
d) 0.088	e) 0.014	f) 9.3048
g) 99.996	h) 0.0048	i) 3.0037

Significant figures

Numbers can also be approximated to a given number of significant figures (s.f.). In the number 43.25 the 4 is the most significant figure as it has a value of 40. In contrast, the 5 is the least significant as it only has a value of 5 hundredths.

Worked examples

a) Write 43.25 to 3 s.f.
Only the three most significant digits are written, however the fourth digit needs to be considered to see whether the third digit is to be rounded up or not.

i.e. 43.25 is written as 43.3 to 3 s.f.

b) Write 0.0043 to 1 s.f.
In this example only two digits have any significance, the 4 and the 3. The 4 is the most significant and therefore is the only one of the two to be written in the answer.

i.e. 0.0043 is written as 0.004 to 1 s.f.

Exercise 2.3

1. Write the following to the number of significant figures written in brackets:

a) 48 599 (1 s.f.)	b) 48 599 (3 s.f.)	c) 6841 (1 s.f.)
d) 7538 (2 s.f.)	e) 483.7 (1 s.f.)	f) 2.5728 (3 s.f.)
g) 990 (1 s.f.)	h) 2045 (2 s.f.)	i) 14.952 (3 s.f.)

2. Write the following to the number of significant figures written in brackets:

a) 0.085 62 (1 s.f.)	b) 0.5932 (1 s.f.)	c) 0.942 (2 s.f.)
d) 0.954 (1 s.f.)	e) 0.954 (2 s.f.)	f) 0.003 05 (1 s.f.)
g) 0.003 05 (2 s.f.)	h) 0.009 73 (2 s.f.)	i) 0.009 73 (1 s.f.)

Appropriate accuracy

In many instances calculations carried out using a calculator produce answers which are not whole numbers. A calculator will give the answer to as many decimal places as will fit on its screen. In most cases this degree of accuracy is neither desirable nor necessary. Unless specifically asked for, answers should not be given to more than two decimal places. Indeed, one decimal place is usually sufficient. In the examination, you will usually be asked to give your answers exactly or correct to three significant figures as appropriate; answers in degrees to be given to one decimal place.

Worked example

Calculate $4.64 \div 2.3$ giving your answer to an appropriate degree of accuracy.

The calculator will give the answer to $4.64 \div 2.3$ as 2.0173913. However the answer given to 1 d.p. is sufficient.

Therefore $4.64 \div 2.3 = 2.0$ (1 d.p.).

Estimating answers to calculations

Even though many calculations can be done quickly and effectively on a calculator, often an estimate for an answer can be a useful check. This is done by rounding each of the numbers in such a way that the calculation becomes relatively straightforward.

Worked examples

a) Estimate the answer to 57×246.

Here are two possibilities:

i) $60 \times 200 = 12\,000$,

ii) $50 \times 250 = 12\,500$.

b) Estimate the answer to $6386 \div 27$.

$6000 \div 30 = 200$.

Exercise 2.4

1. Calculate the following, giving your answer to an appropriate degree of accuracy:

a) 23.456×17.89 b) 0.4×12.62 c) 18×9.24

d) $76.24 \div 3.2$ e) 7.6^2 f) 16.42^3

g) $\dfrac{2.3 \times 3.37}{4}$ h) $\dfrac{8.31}{2.02}$ i) $9.2 \div 4^2$

2. Without using a calculator, estimate the answers to the following:

a) 62×19 b) 270×12 c) 55×60

d) 4950×28 e) 0.8×0.95 f) 0.184×475

3. Without using a calculator, estimate the answers to the following:

a) $3946 \div 18$ b) $8287 \div 42$ c) $906 \div 27$

d) $5520 \div 13$ e) $48 \div 0.12$ f) $610 \div 0.22$

4. Without using a calculator, estimate the answers to the following:

a) $78.45 + 51.02$ b) $168.3 - 87.09$ c) 2.93×3.14

d) $84.2 \div 19.5$ e) $\dfrac{4.3 \times 752}{15.6}$ f) $\dfrac{(9.8)^3}{(2.2)^2}$

5. Using estimation, identify which of the following are definitely incorrect. Explain your reasoning clearly.

a) $95 \times 212 = 20\,140$ b) $44 \times 17 = 748$

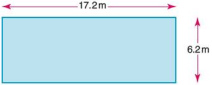
c) $689 \times 413 = 28\,457$ d) $142\,656 \div 8 = 17\,832$

e) $77.9 \times 22.6 = 2512.54$

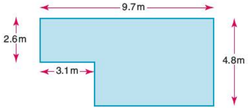
f) $\dfrac{8.42 \times 46}{0.2} = 19\,366$

6. Estimate the shaded areas of the following shapes. Do *not* work out an exact answer.

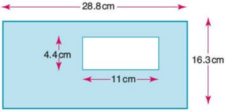
a)



b)

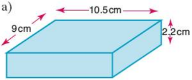


c)

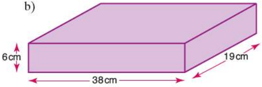


7. Estimate the volume of each of the solids below. Do *not* work out an exact answer.

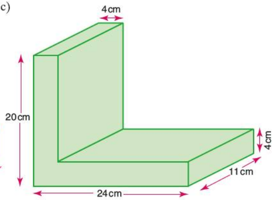
a)



b)



c)



Upper and lower bounds

Numbers can be written to different degrees of accuracy. For example 4.5, 4.50 and 4.500, although appearing to represent the same number, do not. This is because they are written to different degrees of accuracy.

4.5 is rounded to one decimal place and therefore any number from 4.45 up to but not including 4.55 would be rounded to 4.5. On a number line this would be represented as:

As an inequality where x represents the number it would be expressed as

$$4.45 \leqslant x < 4.55$$

4.45 is known as the **lower bound** of 4.5, whilst 4.55 is known as the **upper bound**.

Note that ○——► implies that the number is not included in the solution whilst ●——► implies that the number is included in the solution.

4.50 on the other hand is written to two decimal places and therefore only numbers from 4.495 up to but not including 4.505 would be rounded to 4.50. This therefore represents a much smaller range of numbers than that being rounded to 4.5. Similarly the range of numbers being rounded to 4.500 would be even smaller.

Worked example A girl's height is given as 162 cm to the nearest centimetre.

i) Work out the lower and upper bounds within which her height can lie.

Lower bound = 161.5 cm
Upper bound = 162.5 cm

ii) Represent this range of numbers on a number line.

iii) If the girl's height is h cm, express this range as an inequality.

$$161.5 \leqslant h < 162.5$$

Exercise 2.5

1. Each of the following numbers is expressed to the nearest whole number.
   i) Give the upper and lower bounds of each.
   ii) Using x as the number, express the range in which the number lies as an inequality.
      a) 6　　b) 83　　c) 152
      d) 1000　　e) 100

2. Each of the following numbers is correct to one decimal place.
   i) Give the upper and lower bounds of each.
   ii) Using x as the number, express the range in which the number lies as an inequality.
      a) 3.8　　b) 15.6　　c) 1.0
      d) 10.0　　e) 0.3

3. Each of the following numbers is correct to two significant figures.
   i) Give the upper and lower bounds of each.
   ii) Using x as the number, express the range in which the number lies as an inequality.
      a) 4.2　　b) 0.84　　c) 420
      d) 5000　　e) 0.045　　f) 25 000

4. The mass of a sack of vegetables is given as 5.4 kg.
   a) Illustrate the lower and upper bounds of the mass on a number line.
   b) Using M kg for the mass, express the range of values in which M must lie, as an inequality.

5. At a school sports day, the winning time for the 100 m race was given as 11.8 seconds.
   a) Illustrate the lower and upper bounds of the time on a number line.
   b) Using T seconds for the time, express the range of values in which T must lie, as an inequality.

6. The capacity of a swimming pool is given as 620 m^3 correct to two significant figures.
   a) Calculate the lower and upper bounds of the pool's capacity.
   b) Using x cubic metres for the capacity, express the range of values in which x must lie, as an inequality.

7. A farmer measures the dimensions of his rectangular field to the nearest 10 m. The length is recorded as 630 m and the width is recorded as 400 m.
   a) Calculate the lower and upper bounds of the length.
   b) Using W metres for the width, express the range of values in which W must lie, as an inequality.

Calculating with upper and lower bounds

When numbers are written to a specific degree of accuracy, calculations involving those numbers also give a range of possible answers.

Worked examples

a) Calculate the upper and lower bounds for the following calculation, given that each number is given to the nearest whole number.

34×65

34 lies in the range $33.5 \leqslant x < 34.5$.
65 lies in the range $64.5 \leqslant x < 65.5$.

The lower bound of the calculation is obtained by multiplying together the two lower bounds. Therefore the minimum product is 33.5×64.5, i.e. 2160.75.

The upper bound of the calculation is obtained by multiplying together the two upper bounds. Therefore the maximum product is 34.5×65.5, i.e. 2259.75.

b) Calculate the upper and lower bounds to $\frac{33.5}{22.0}$ given that each of the numbers is accurate to 1 d.p.

33.5 lies in the range $33.45 \leqslant x < 33.55$.
22.0 lies in the range $21.95 \leqslant x < 22.05$.

The lower bound of the calculation is obtained by dividing the lower bound of the numerator by the *upper* bound of the denominator. So the minimum value is $33.45 \div 22.05$, i.e. 1.52 (2 d.p.).

The upper bound of the calculation is obtained by dividing the upper bound of the numerator by the *lower* bound of the denominator. So the maximum value is $33.55 \div 21.95$, i.e. 1.53 (2 d.p.).

Exercise 2.6

1. Calculate lower and upper bounds for the following calculations, if each of the numbers is given to the nearest whole number.

a) 14×20 b) 135×25 c) 100×50

d) $\frac{40}{10}$ e) $\frac{33}{11}$ f) $\frac{125}{15}$

g) $\frac{12 \times 65}{16}$ h) $\frac{101 \times 28}{69}$ i) $\frac{250 \times 7}{100}$

j) $\frac{44}{3^2}$ k) $\frac{578}{17 \times 22}$ l) $\frac{1000}{4 \times (3+8)}$

2. Calculate lower and upper bounds for the following calculations, if each of the numbers is given to 1 d.p.

a) $2.1 + 4.7$ b) 6.3×4.8 c) 10.0×14.9

d) $17.6 - 4.2$ e) $\frac{8.5+3.6}{6.8}$ f) $\frac{7.7-6.2}{3.5}$

g) $\frac{(16.4)^2}{(3.0-0.3)^2}$ h) $\frac{100.0}{(50.0-40.0)^2}$ i) $(0.1 - 0.2)^2$

3. Calculate lower and upper bounds for the following calculations, if each of the numbers is given to 2 s.f.

a) 64×320 b) 1.7×0.65 c) 4800×240

d) $\frac{54\,000}{600}$ e) $\frac{4.2}{0.031}$ f) $\frac{100}{5.2}$

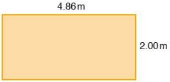
g) $\frac{6.8 \times 42}{120}$ h) $\frac{100}{(4.5 \times 6.0)}$ i) $\frac{180}{(7.3-4.5)}$

Exercise 2.7

1. The masses to the nearest 0.5 kg of two parcels are 1.5 kg and 2.5 kg. Calculate the lower and upper bounds of their combined mass.

2. Calculate upper and lower bounds for the perimeter of the rectangle shown (below), if its dimensions are correct to 1 d.p.

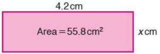
3. Calculate upper and lower bounds for the perimeter of the rectangle shown (below), whose dimensions are accurate to 2 d.p.



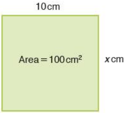
4. Calculate upper and lower bounds for the area of the rectangle shown (below), if its dimensions are accurate to 1 d.p.

5. Calculate upper and lower bounds for the area of the rectangle shown (below), whose dimensions are correct to 2 s.f.

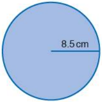
6. Calculate upper and lower bounds for the length marked x cm in the rectangle (below). The area and length are both given to 1 d.p.



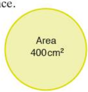
7. Calculate the upper and lower bounds for the length marked x cm in the rectangle (below). The area and length are both accurate to 2 s.f.



8. The radius of the circle shown (below) is given to 1 d.p. Calculate the upper and lower bounds of:
   a) the circumference,
   b) the area.



9. The area of the circle shown (below) is given to 2 s.f. Calculate the upper and lower bounds of:
   a) the radius,
   b) the circumference.



10. The mass of a cube of side 2 cm is given as 100 g. The side is accurate to the nearest millimetre and the mass accurate to the nearest gram. Calculate the maximum and minimum possible values for the density of the material (density = mass ÷ volume).

11. The distance to the nearest 100 000 km from Earth to the moon is given as 400 000 km. The average speed to the nearest 500 km/h of a rocket to the moon is given as 3500 km/h. Calculate the greatest and least time it could take the rocket to reach the moon.

Student assessment 1

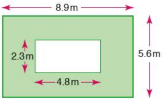
1. Round the following numbers to the degree of accuracy shown in brackets:
a) 2841 (nearest 100)
b) 7286 (nearest 10)
c) 48 756 (nearest 1000)
d) 951 (nearest 100)

2. Round the following numbers to the number of decimal places shown in brackets:
a) 3.84 (1 d.p.)
b) 6.792 (1 d.p.)
c) 0.8526 (2 d.p.)
d) 1.5849 (2 d.p.)
e) 9.954 (1 d.p.)
f) 0.0077 (3 d.p.)

3. Round the following numbers to the number of significant figures shown in brackets:
a) 3.84 (1 s.f.)
b) 6.792 (2 s.f.)
c) 0.7765 (1 s.f.)
d) 9.624 (1 s.f.)
e) 834.97 (2 s.f.)
f) 0.004 51 (1 s.f.)

4. 1 mile is 1760 yards. Estimate the number of yards in 11.5 miles.

5. Estimate the shaded area of the figure below:



6. Estimate the answers to the following. Do *not* work out an exact answer.

a) $\dfrac{5.3 \times 11.2}{2.1}$ b) $\dfrac{(9.8)^2}{(4.7)^2}$ c) $\dfrac{18.8 \times (7.1)^2}{(3.1)^2 \times (4.9)^2}$

7. A cuboid's dimensions are given as 12.32 cm by 1.8 cm by 4.16 cm. Calculate its volume, giving your answer to an appropriate degree of accuracy.

Student assessment 2

1. The following numbers are expressed to the nearest whole number. Illustrate on a number line the range in which each must lie.

a) 7 b) 40 c) 300 d) 2000

2. The following numbers are expressed correct to two significant figures. Representing each number by the letter x, express the range in which each must lie, using an inequality.

a) 210 b) 64 c) 3.0 d) 0.88

3. A school measures the dimensions of its rectangular playing field to the nearest metre. The length was recorded as 350 m and the width as 200 m. Express the range in which the length and width lie using inequalities.

4. A boy's mass was measured to the nearest 0.1 kg. If his mass was recorded as 58.9 kg, illustrate on a number line the range within which it must lie.

5. An electronic clock is accurate to $\frac{1}{1000}$ of a second. The duration of a flash from a camera is timed at 0.004 seconds. Express the upper and lower bounds of the duration of the flash using inequalities.

6. The following numbers are rounded to the degree of accuracy shown in brackets. Express the lower and upper bounds of these numbers as an inequality.

a) $x = 4.83$ (2 d.p.) b) $y = 5.05$ (2 d.p.)
c) $z = 10.0$ (1 d.p.) d) $p = 100.00$ (2 d.p.)

Student assessment 3

1. Calculate the upper and lower bounds of the following calculations given that each number is written to the nearest whole number.

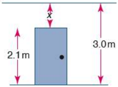
a) 20×50 b) 100×63 c) $\frac{500}{80}$

d) $\frac{14 \times 73}{20}$ e) $\frac{17-7}{4+6}$ f) $\frac{8\times(3+6)}{10^2}$

2. In the rectangle (below) both dimensions are given to 1 d.p. Calculate the upper and lower bounds for the area.

3. An equilateral triangle has sides of length 4 cm correct to the nearest whole number. Calculate the upper and lower bounds for the perimeter of the triangle.

4. The height to 1 d.p. of a room is given as 3.0 m. A door to the room has a height to 1 d.p. of 2.1 m. Write as an inequality the upper and lower bounds for the gap between the top of the door and the ceiling.



5. The mass of 85 oranges is given as 40 kg correct to 2 s.f. Calculate the lower and upper bounds for the average mass of one orange.

Student assessment 4

1. Five boys have a mass, given to the nearest 10 kg, of: 40 kg, 50 kg, 50 kg, 60 kg and 80 kg. Calculate the least possible total mass.

2. A water tank measures 30 cm by 50 cm by 20 cm. If each of these measurements is given to the nearest centimetre, calculate the largest possible volume of the tank.

3. The volume of a cube is given as 125 cm^3 to the nearest whole number.
   a) Express as an inequality the upper and lower bounds of the cube's volume.
   b) Express as an inequality the upper and lower bounds of the length of each of the cube's edges.

4. The radius of a circle is given as 4.00 cm to 2 d.p. Express as an inequality the upper and lower bounds for:
   a) the circumference of the circle,
   b) the area of the circle.

5. A cylindrical water tank has a volume of 6000 cm^3 correct to 1 s.f. A full cup of water from the tank has a volume of 300 cm^3 correct to 2 s.f. Calculate the maximum number of full cups of water that can be drawn from the tank.

6. A match measures 5 cm to the nearest centimetre. 100 matches end to end measure 5.43 m correct to 3 s.f.
   a) Calculate the upper and lower limits of the length of one match.
   b) How can the limits of the length of a match be found to 2 d.p.?

3 Calculations and order

Ordering

The following symbols have a specific meaning in mathematics:

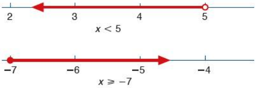
- $=$ is equal to
- $\neq$ is not equal to
- $>$ is greater than
- $\geqslant$ is greater than or equal to
- $<$ is less than
- $\leqslant$ is less than or equal to

$x \geqslant 3$ implies that x is greater than or equal to 3, i.e. x can be 3, 4, 4.2, 5, 5.6, etc.
$3 \leqslant x$ implies that 3 is less than or equal to x, i.e. x is still 3, 4, 4.2, 5, 5.6, etc.

Therefore:

$5 > x$ can be rewritten as $x < 5$, i.e. x can be 4, 3.2, 3, 2.8, 2, 1, etc.
$-7 \leqslant x$ can be rewritten as $x \geqslant -7$, i.e. x can be -7, -6, -5, etc.

These inequalities can also be represented on a number line:



Note that  implies that the number is not included in the solution whilst  implies that the number is included in the solution.

Worked examples

a) The maximum number of players from one football team allowed on the pitch at any one time is eleven. Represent this information:
i) as an inequality,
ii) on a number line.

i) Let the number of players be represented by the letter n. n must be less than or equal to 11. Therefore $n \leqslant 11$.

ii) 

b) The maximum number of players from one football team allowed on the pitch at any one time is eleven. The minimum allowed is seven players. Represent this information:

i) as an inequality,
ii) on a number line.

i) Let the number of players be represented by the letter n. n must be greater than or equal to 7, but less than or equal to 11.

Therefore $7 \leqslant n \leqslant 11$.

ii) 

Exercise 3.1

1. Copy each of the following statements, and insert one of the symbols =, >, < into the space to make the statement correct:

a) $7 \times 2 \ldots 8 + 7$ b) $6^2 \ldots 9 \times 4$
c) $5 \times 10 \ldots 7^2$ d) 80 cm . . . 1 m
e) 1000 litres . . . 1 m^3 f) $48 \div 6 \ldots 54 \div 9$

2. Represent each of the following inequalities on a number line, where x is a real number:

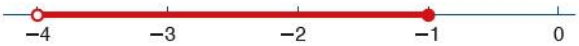
a) $x < 2$ b) $x \geqslant 3$
c) $x \leqslant -4$ d) $x \geqslant -2$
e) $2 < x < 5$ f) $-3 < x < 0$
g) $-2 \leqslant x < 2$ h) $2 \geqslant x \geqslant -1$

3. Write down the inequalities which correspond to the following number lines:

a) 

b) 

c) 

d) 

4. Write the following sentences using inequality signs.

a) The maximum capacity of an athletics stadium is 20 000 people.
b) In a class the tallest student is 180 cm and the shortest is 135 cm.
c) Five times a number plus 3 is less than 20.
d) The maximum temperature in May was 25 °C.
e) A farmer has between 350 and 400 apples on each tree in his orchard.
f) In December temperatures in Kenya were between 11 °C and 28 °C.

Exercise 3.2

1. Write the following decimals in order of magnitude, starting with the smallest:

   6.0 0.6 0.66 0.606 0.06 6.6 6.606

2. Write the following fractions in order of magnitude, starting with the largest:

   $\frac{1}{2}$ $\frac{1}{3}$ $\frac{6}{13}$ $\frac{4}{5}$ $\frac{7}{18}$ $\frac{2}{19}$

3. Write the following lengths in order of magnitude, starting with the smallest:

   2 m 60 cm 800 mm 180 cm 0.75 m

4. Write the following masses in order of magnitude, starting with the largest:

   4 kg 3500 g $\frac{3}{4}$ kg 700 g 1 kg

5. Write the following volumes in order of magnitude, starting with the smallest:

   1 l 430 ml 800 cm^3 120 cl 150 cm^3

● The order of operations

When carrying out calculations, care must be taken to ensure that they are carried out in the correct order.

Worked examples

a) Use a scientific calculator to work out the answer to the following:

$2 + 3 \times 4 =$

[2] [+] [3] [×] [4] [=] 14

b) Use a scientific calculator to work out the answer to the following:

$(2 + 3) \times 4 =$

[(] [2] [+] [3] [)] [×] [4] [=] 20

The reason why different answers are obtained is because, by convention, the operations have different priorities. These are as follows:

(1) brackets
(2) multiplication/division
(3) addition/subtraction

Therefore in ***Worked example* a)** 3×4 is evaluated first, and then the 2 is added, whilst in ***Worked example* b)** $(2 + 3)$ is evaluated first, followed by multiplication by 4.

Exercise 3.3

In the following questions, evaluate the answers:

i) in your head,
ii) using a scientific calculator.

1. a) $8 \times 3 + 2$ b) $4 \div 2 + 8$
c) $12 \times 4 - 6$ d) $4 + 6 \times 2$
e) $10 - 6 \div 3$ f) $6 - 3 \times 4$

2. a) $7 \times 2 + 3 \times 2$ b) $12 \div 3 + 6 \times 5$
c) $9 + 3 \times 8 - 1$ d) $36 - 9 \div 3 - 2$
e) $14 \times 2 - 16 \div 2$ f) $4 + 3 \times 7 - 6 \div 3$

3. a) $(4 + 5) \times 3$ b) $8 \times (12 - 4)$
c) $3 \times (8 + 3) - 3$ d) $(4 + 11) \div (7 - 2)$
e) $4 \times 3 \times (7 + 5)$ f) $24 \div 3 \div (10 - 5)$

Exercise 3.4

In each of the following questions:

i) Copy the calculation and put in any brackets which are needed to make it correct.
ii) Check your answer using a scientific calculator.

1. a) $6 \times 2 + 1 = 18$ b) $1 + 3 \times 5 = 16$
c) $8 + 6 \div 2 = 7$ d) $9 + 2 \times 4 = 44$
e) $9 \div 3 \times 4 + 1 = 13$ f) $3 + 2 \times 4 - 1 = 15$

2. a) $12 \div 4 - 2 + 6 = 7$ b) $12 \div 4 - 2 + 6 = 12$
c) $12 \div 4 - 2 + 6 = -5$ d) $12 \div 4 - 2 + 6 = 1.5$
e) $4 + 5 \times 6 - 1 = 33$ f) $4 + 5 \times 6 - 1 = 29$
g) $4 + 5 \times 6 - 1 = 53$ h) $4 + 5 \times 6 - 1 = 45$

It is important to use brackets when dealing with more complex calculations.

Worked examples

a) Evaluate the following using a scientific calculator:

$$\frac{12+9}{10-3} =$$

(1 2 + 9) ÷ (1 0 − 3) = 3

b) Evaluate the following using a scientific calculator:

$$\frac{20+12}{4^2} =$$

(2 0 + 1 2) ÷ 4 x^2 = 2

c) Evaluate the following using a scientific calculator:

$$\frac{90+38}{4^3}=$$

[( ] [9] [0] [+] [3] [8] [ )] [÷] [4] [x^y] [3] [=]   2

Note: different types of calculator have different 'to the power of' buttons.

Exercise 3.5

Using a scientific calculator, evaluate the following:

1. a) $\dfrac{9+3}{6}$ b) $\dfrac{30-6}{5+3}$

c) $\dfrac{40+9}{12-5}$ d) $\dfrac{15\times 2}{7+8}+2$

e) $\dfrac{100+21}{11}+4\times 3$ f) $\dfrac{7+2\times 4}{7-2}-3$

2. a) $\dfrac{4^2-6}{2+8}$ b) $\dfrac{3^2+4^2}{5}$

c) $\dfrac{6^3-4^2}{4\times 25}$ d) $\dfrac{3^3\times 4^4}{12^2}+2$

e) $\dfrac{3+3^3}{5}+\dfrac{4^2-2^3}{8}$ f) $\dfrac{(6+3)\times 4}{2^3}-2\times 3$

Student assessment 1

1. Write the information on the following number lines as inequalities:

a) b) c) d)

2. Illustrate each of the following inequalities on a number line:
   a) $x \geqslant 3$ b) $x < 4$
   c) $0 < x < 4$ d) $-3 \leqslant x < 1$

3. Write the following fractions in order of magnitude, starting with the smallest:

$$\frac{4}{7} \quad \frac{3}{14} \quad \frac{9}{10} \quad \frac{1}{2} \quad \frac{2}{5}$$

Student assessment 2

1. Evaluate the following:
   a) $6 \times 8 - 4$ b) $3 + 5 \times 2$
   c) $3 \times 3 + 4 \times 4$ d) $3 + 3 \times 4 + 4$
   e) $(5 + 2) \times 7$ f) $18 \div 2 \div (5 - 2)$

2. Copy the following, if necessary putting in brackets to make the statement correct:
   a) $7 - 4 \times 2 = 6$ b) $12 + 3 \times 3 + 4 = 33$
   c) $5 + 5 \times 6 - 4 = 20$ d) $5 + 5 \times 6 - 4 = 56$

3. Evaluate the following using a calculator:
   a) $\dfrac{2^4 - 3^2}{2}$ b) $\dfrac{(8-3) \times 3}{5} + 7$

Student assessment 3

1. Evaluate the following:
   a) $3 \times 9 - 7$ b) $12 + 6 \div 2$
   c) $3 + 4 \div 2 \times 4$ d) $6 + 3 \times 4 - 5$
   e) $(5 + 2) \div 7$ f) $14 \times 2 \div (9 - 2)$

2. Copy the following, if necessary putting in brackets to make the statement correct:
   a) $7 - 5 \times 3 = 6$ b) $16 + 4 \times 2 + 4 = 40$
   c) $4 + 5 \times 6 - 1 = 45$ d) $1 + 5 \times 6 - 6 = 30$

3. Using a calculator, evaluate the following:
   a) $\dfrac{3^3 - 4^2}{2}$ b) $\dfrac{(15-3) \div 3}{2} + 7$

4 Integers, fractions, decimals and percentages

Fractions

A single unit can be broken into equal parts called fractions, e.g. $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$.

If, for example, the unit is broken into ten equal parts and three parts are then taken, the fraction is written as $\frac{3}{10}$. That is, three parts out of ten parts.

In the fraction $\frac{3}{10}$:

The ten is called the **denominator**.

The three is called the **numerator**.

A **proper fraction** has its numerator less than its denominator, e.g. $\frac{3}{4}$.

An **improper fraction** has its numerator more than its denominator, e.g. $\frac{9}{2}$.

A **mixed number** is made up of a whole number and a proper fraction, e.g. $4\frac{1}{5}$.

Worked examples

a) Find $\frac{1}{5}$ of 35.

This means 'divide 35 into 5 equal parts'.

$\frac{1}{5}$ of 35 is $35 \div 5 = 7$.

b) Find $\frac{3}{5}$ of 35.

Since $\frac{1}{5}$ of 35 is 7, $\frac{3}{5}$ of 35 is 7×3.

That is, 21.

Exercise 4.1

1. Evaluate the following:

a) $\frac{3}{4}$ of 12 b) $\frac{4}{5}$ of 20 c) $\frac{4}{9}$ of 45 d) $\frac{5}{8}$ of 64

e) $\frac{3}{11}$ of 66 f) $\frac{9}{10}$ of 80 g) $\frac{5}{7}$ of 42 h) $\frac{8}{9}$ of 54

i) $\frac{7}{8}$ of 240 j) $\frac{4}{5}$ of 65

Changing a mixed number to an improper fraction

Worked examples

a) Change $3\frac{5}{8}$ into an improper fraction.

$$3\frac{5}{8} = \frac{24}{8} + \frac{5}{8}$$
$$= \frac{24+5}{8}$$
$$= \frac{29}{8}$$

b) Change $\frac{27}{4}$ into a mixed number.

$$\frac{27}{4} = \frac{24+3}{4}$$
$$= \frac{24}{4} + \frac{3}{4}$$
$$= 6\frac{3}{4}$$

Exercise 4.2

1. Change the following mixed numbers into improper fractions:

a) $4\frac{2}{3}$	b) $3\frac{3}{5}$	c) $5\frac{7}{8}$	d) $2\frac{5}{6}$
e) $8\frac{1}{2}$	f) $9\frac{5}{7}$	g) $6\frac{4}{9}$	h) $4\frac{1}{4}$
i) $5\frac{4}{11}$	j) $7\frac{6}{7}$	k) $4\frac{3}{10}$	l) $11\frac{3}{13}$

2. Change the following improper fractions into mixed numbers.

a) $\frac{29}{4}$	b) $\frac{33}{5}$	c) $\frac{41}{6}$	d) $\frac{53}{8}$
e) $\frac{49}{9}$	f) $\frac{17}{12}$	g) $\frac{66}{7}$	h) $\frac{33}{10}$
i) $\frac{19}{2}$	j) $\frac{73}{12}$		

Decimals

H	T	U.	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
		3.	2	7	
		0.	0	3	8

3.27 is 3 units, 2 tenths and 7 hundredths

i.e. $3.27 = 3 + \frac{2}{10} + \frac{7}{100}$

0.038 is 3 hundredths and 8 thousandths

i.e. $0.038 = \frac{3}{100} + \frac{8}{1000}$

Note that 2 tenths and 7 hundredths is equivalent to 27 hundredths

i.e. $\frac{2}{10} + \frac{7}{100} = \frac{27}{100}$

and that 3 hundredths and 8 thousandths is equivalent to 38 thousandths

i.e. $\frac{3}{100} + \frac{8}{1000} = \frac{38}{1000}$

Exercise 4.3

1. Write the following fractions as decimals:

a) $4\frac{5}{10}$	b) $6\frac{3}{10}$	c) $17\frac{8}{10}$	d) $3\frac{7}{100}$
e) $9\frac{27}{100}$	f) $11\frac{36}{100}$	g) $4\frac{6}{1000}$	h) $5\frac{27}{1000}$
i) $4\frac{356}{1000}$	j) $9\frac{204}{1000}$		

2. Evaluate the following without using a calculator:

a) $2.7 + 0.35 + 16.09$ b) $1.44 + 0.072 + 82.3$
c) $23.8 - 17.2$ d) $16.9 - 5.74$
e) $121.3 - 85.49$ f) $6.03 + 0.5 - 1.21$
g) $72.5 - 9.08 + 3.72$ h) $100 - 32.74 - 61.2$
i) $16.0 - 9.24 - 5.36$ j) $1.1 - 0.92 - 0.005$

Percentages

A fraction whose denominator is 100 can be expressed as a percentage.

$\frac{29}{100}$ can be written as 29%

$\frac{45}{100}$ can be written as 45%

By using equivalent fractions to change the denominator to 100, other fractions can be written as percentages.

Worked example

Change $\frac{3}{5}$ to a percentage.

$\frac{3}{5} = \frac{3}{5} \times \frac{20}{20} = \frac{60}{100}$

$\frac{60}{100}$ can be written as 60%

Exercise 4.4

1. Express each of the following as a fraction with denominator 100, then write them as percentages:

a) $\frac{29}{50}$ b) $\frac{17}{25}$ c) $\frac{11}{20}$ d) $\frac{3}{10}$
e) $\frac{23}{25}$ f) $\frac{19}{50}$ g) $\frac{3}{4}$ h) $\frac{2}{5}$

2. Copy and complete the table of equivalents below.

Fraction	Decimal	Percentage
$\frac{1}{10}$		
	0.2	
		30%
$\frac{4}{10}$		
	0.5	
		60%
	0.7	
$\frac{4}{5}$		
	0.9	
$\frac{1}{4}$		
		75%

The four rules

Addition, subtraction, multiplication and division are mathematical operations.

Long multiplication

When carrying out long multiplication, it is important to remember place value.

Worked example 184×37

$$
\begin{array}{rrrrl}
 & 1 & 8 & 4 & \\
\times & & 3 & 7 & \\
\hline
1 & 2 & 8 & 8 & (184 \times 7) \\
5 & 5 & 2 & 0 & (184 \times 30) \\
\hline
6 & 8 & 0 & 8 & (184 \times 37)
\end{array}
$$

Short division

Worked example $453 \div 6$

$$
\begin{array}{r|l}
 & \;\;7\;\;5\;\text{r}3 \\
\hline
6 & 4\;\;5\;\;{}^{3}3
\end{array}
$$

It is usual, however, to give the final answer in decimal form rather than with a remainder. The division should therefore be continued:

$453 \div 6$

$$
\begin{array}{r|l}
 & \;\;7\;\;5\,.\,5 \\
\hline
6 & 4\;\;5\;\;{}^{3}3\,.\,{}^{3}0
\end{array}
$$

Long division

Worked example Calculate $7184 \div 23$ to one decimal place (1 d.p.).

$$
\begin{array}{r|rrrrrr}
 & & 3 & 1 & 2. & 3 & 4 \\
\hline
23 & 7 & 1 & 8 & 4. & 0 & 0 \\
 & 6 & 9 & & & & \\
\cline{2-3}
 & & 2 & 8 & & & \\
 & & 2 & 3 & & & \\
\cline{3-4}
 & & & 5 & 4 & & \\
 & & & 4 & 6 & & \\
\cline{4-5}
 & & & & 8 & 0 & \\
 & & & & 6 & 9 & \\
\cline{5-6}
 & & & & 1 & 1 & 0 \\
 & & & & & 9 & 2 \\
\cline{6-7}
 & & & & & 1 & 8 \\
\cline{6-7}
\end{array}
$$

Therefore $7184 \div 23 = 312.3$ to 1 d.p.

Mixed operations

When a calculation involves a mixture of operations, the order of the operations is important. Multiplications and divisions are done first, whilst additions and subtractions are done afterwards. To override this, brackets need to be used.

Worked examples

a) $3 + 7 \times 2 - 4$
$= 3 + 14 - 4$
$= 13$

b) $(3 + 7) \times 2 - 4$
$= 10 \times 2 - 4$
$= 20 - 4$
$= 16$

c) $3 + 7 \times (2 - 4)$
$= 3 + 7 \times (-2)$
$= 3 - 14$
$= -11$

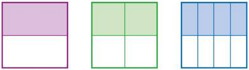
d) $(3 + 7) \times (2 - 4)$
$= 10 \times (-2)$
$= -20$

Exercise 4.5

1. Evaluate the answer to each of the following:
   a) $3 + 5 \times 2 - 4$ b) $12 \div 8 + 6 \div 4$
2. Copy these equations and put brackets in the correct places to make them correct:
   a) $6 \times 4 + 6 \div 3 = 20$ b) $9 - 3 \times 7 + 2 = 54$
3. Without using a calculator, work out the solutions to the following multiplications:
   a) 785×38 b) 164×253
4. Work out the remainders in the following divisions:
   a) $72 \div 7$ b) $430 \div 9$
5. a) A length of rail track is 9 m long. How many complete lengths will be needed to lay 1 km of track?
   b) How many 35 cent stamps can be bought for 10 dollars?
6. Work out the following long divisions to 1 d.p.
   a) $7892 \div 7$ b) $7892 \div 15$

● Calculations with fractions

Equivalent fractions



$\frac{1}{2}$ $\frac{2}{4}$ $\frac{4}{8}$

It should be apparent that $\frac{1}{2}$, $\frac{2}{4}$ and $\frac{4}{8}$ are equivalent fractions.

Similarly, $\frac{1}{3}$, $\frac{2}{6}$, $\frac{3}{9}$ and $\frac{4}{12}$ are equivalent, as are $\frac{1}{5}$, $\frac{10}{50}$ and $\frac{20}{100}$. Equivalent fractions are mathematically the same as each other. In the diagrams above $\frac{1}{2}$ is mathematically the same as $\frac{4}{8}$. However $\frac{1}{2}$ is a simplified form of $\frac{4}{8}$.

When carrying out calculations involving fractions it is usual to give your answer in its **simplest form**. Another way of saying 'simplest form' is '**lowest terms**'.

Worked examples

a) Write $\frac{4}{22}$ in its simplest form.

Divide both the numerator and the denominator by their highest common factor.

The highest common factor of both 4 and 22 is 2.

Dividing both 4 and 22 by 2 gives $\frac{2}{11}$.

Therefore $\frac{2}{11}$ is $\frac{4}{22}$ written in its simplest form.

b) Write $\frac{12}{40}$ in its lowest terms.

Divide both the numerator and the denominator by their highest common factor.

The highest common factor of both 12 and 40 is 4.

Dividing both 12 and 40 by 4 gives $\frac{3}{10}$.

Therefore $\frac{3}{10}$ is $\frac{12}{40}$ written in its lowest terms.

Exercise 4.6

1. Express the following fractions in their lowest terms.
e.g. $\frac{12}{16} = \frac{3}{4}$

a) $\frac{5}{10}$ b) $\frac{7}{21}$ c) $\frac{8}{12}$

d) $\frac{16}{36}$ e) $\frac{75}{100}$ f) $\frac{81}{90}$

Addition and subtraction of fractions

For fractions to be either added or subtracted, the denominators need to be the same.

Worked examples

a) $\frac{3}{11} + \frac{5}{11} = \frac{8}{11}$

b) $\frac{7}{8} + \frac{5}{8} = \frac{12}{8} = 1\frac{1}{2}$

c) $\frac{1}{2} + \frac{1}{3}$
$= \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$

d) $\frac{4}{5} - \frac{1}{3}$
$= \frac{12}{15} - \frac{5}{15} = \frac{7}{15}$

When dealing with calculations involving mixed numbers, it is sometimes easier to change them to improper fractions first.

Worked examples

a) $5\frac{3}{4} - 2\frac{5}{8}$
$= \frac{23}{4} - \frac{21}{8}$
$= \frac{46}{8} - \frac{21}{8}$
$= \frac{25}{8} = 3\frac{1}{8}$

b) $1\frac{4}{7} + 3\frac{3}{4}$
$= \frac{11}{7} + \frac{15}{4}$
$= \frac{44}{28} + \frac{105}{28}$
$= \frac{149}{28} = 5\frac{9}{28}$

Exercise 4.7

Evaluate each of the following and write the answer as a fraction in its simplest form:

1. a) $\frac{3}{5} + \frac{4}{5}$ b) $\frac{3}{11} + \frac{7}{11}$

c) $\frac{2}{3} + \frac{1}{4}$ d) $\frac{3}{5} + \frac{4}{9}$

e) $\frac{8}{13} + \frac{2}{5}$ f) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4}$

2. a) $\frac{1}{8}+\frac{3}{8}+\frac{5}{8}$ b) $\frac{3}{7}+\frac{5}{7}+\frac{4}{7}$

c) $\frac{1}{3}+\frac{1}{2}+\frac{1}{4}$ d) $\frac{1}{5}+\frac{1}{3}+\frac{1}{4}$

e) $\frac{3}{8}+\frac{3}{5}+\frac{3}{4}$ f) $\frac{3}{13}+\frac{1}{4}+\frac{1}{2}$

3. a) $\frac{3}{7}-\frac{2}{7}$ b) $\frac{4}{5}-\frac{7}{10}$

c) $\frac{8}{9}-\frac{1}{3}$ d) $\frac{7}{12}-\frac{1}{2}$

e) $\frac{5}{8}-\frac{2}{5}$ f) $\frac{3}{4}-\frac{2}{5}+\frac{7}{10}$

4. a) $\frac{3}{4}+\frac{1}{5}-\frac{2}{3}$ b) $\frac{3}{8}+\frac{7}{11}-\frac{1}{2}$

c) $\frac{4}{5}-\frac{3}{10}+\frac{7}{20}$ d) $\frac{9}{13}+\frac{1}{3}-\frac{4}{5}$

e) $\frac{9}{10}-\frac{1}{5}-\frac{1}{4}$ f) $\frac{8}{9}-\frac{1}{3}-\frac{1}{2}$

5. a) $2\frac{1}{2}+3\frac{1}{4}$ b) $3\frac{3}{5}+1\frac{7}{10}$

c) $6\frac{1}{2}-3\frac{2}{5}$ d) $8\frac{5}{8}-2\frac{1}{3}$

e) $5\frac{7}{8}-4\frac{3}{4}$ f) $3\frac{1}{4}-2\frac{5}{9}$

6. a) $2\frac{1}{2}+1\frac{1}{4}+1\frac{3}{8}$ b) $2\frac{4}{5}+3\frac{1}{8}+1\frac{3}{10}$

c) $4\frac{1}{2}-1\frac{1}{4}-3\frac{5}{8}$ d) $6\frac{1}{2}-2\frac{3}{4}-3\frac{2}{5}$

e) $2\frac{4}{7}-3\frac{1}{4}-1\frac{3}{5}$ f) $4\frac{7}{20}-5\frac{1}{2}+2\frac{2}{5}$

Multiplication and division of fractions

Worked examples

a) $\frac{3}{4}\times\frac{2}{3}$

$=\frac{6}{12}$

$=\frac{1}{2}$

b) $3\frac{1}{2}\times 4\frac{4}{7}$

$=\frac{7}{2}\times\frac{32}{7}$

$=\frac{224}{14}$

$=16$

The **reciprocal** of a number is obtained when 1 is divided by that number. The reciprocal of 5 is $\frac{1}{5}$, the reciprocal of $\frac{2}{5}$ is $\frac{5}{2}$, etc.

Dividing fractions is the same as multiplying by the reciprocal.

Worked examples

a) $\frac{3}{8}\div\frac{3}{4}$

$=\frac{3}{8}\times\frac{4}{3}$

$=\frac{12}{24}$

$=\frac{1}{2}$

b) $5\frac{1}{2}\div 3\frac{2}{3}$

$=\frac{11}{2}\div\frac{11}{3}$

$=\frac{11}{2}\times\frac{3}{11}$

$=\frac{3}{2}$

Exercise 4.8

1. Write the reciprocal of each of the following:

a) $\frac{1}{8}$ b) $\frac{7}{12}$ c) $\frac{3}{5}$

d) $1\frac{1}{2}$ e) $3\frac{3}{4}$ f) 6

2. Evaluate the following:

a) $\frac{3}{8} \times \frac{4}{9}$ b) $\frac{2}{3} \times \frac{9}{10}$ c) $\frac{5}{7} \times \frac{4}{15}$

d) $\frac{3}{4}$ of $\frac{8}{9}$ e) $\frac{5}{6}$ of $\frac{3}{10}$ f) $\frac{7}{8}$ of $\frac{2}{5}$

3. Evaluate the following:

a) $\frac{5}{8} \div \frac{3}{4}$ b) $\frac{5}{6} \div \frac{1}{3}$ c) $\frac{4}{5} \div \frac{7}{10}$

d) $1\frac{2}{3} \div \frac{2}{5}$ e) $\frac{3}{7} \div 2\frac{1}{7}$ f) $1\frac{1}{4} \div 1\frac{7}{8}$

4. Evaluate the following:

a) $\frac{3}{4} \times \frac{4}{5}$ b) $\frac{7}{8} \times \frac{2}{3}$

c) $\frac{3}{4} \times \frac{4}{7} \times \frac{3}{10}$ d) $(\frac{4}{5} \div \frac{2}{3}) \times \frac{7}{10}$

e) $\frac{1}{2}$ of $\frac{3}{4}$ f) $4\frac{1}{2} \div 3\frac{1}{9}$

5. Evaluate the following:

a) $(\frac{3}{8} \times \frac{4}{5}) + (\frac{1}{2}$ of $\frac{3}{5})$ b) $(1\frac{1}{2} \times 3\frac{3}{4}) - (2\frac{3}{5} \div 1\frac{1}{2})$

c) $(\frac{3}{5}$ of $\frac{4}{9}) + (\frac{4}{9}$ of $\frac{3}{5})$ d) $(1\frac{1}{3} \times 2\frac{5}{8})^2$

Changing a fraction to a decimal

To change a fraction to a decimal, divide the numerator by the denominator.

Worked examples

a) Change $\frac{5}{8}$ to a decimal.

$$8\,\overline{)\,5.0\,{}^{2}0\,{}^{4}0}\quad = 0.625$$

b) Change $2\frac{3}{5}$ to a decimal.
This can be represented as $2 + \frac{3}{5}$.

$$5\,\overline{)\,3.0}\quad = 0.6$$

Therefore $2\frac{3}{5} = 2.6$

Exercise 4.9

1. Change the following fractions to decimals:

a) $\frac{3}{4}$ b) $\frac{4}{5}$ c) $\frac{9}{20}$

d) $\frac{17}{50}$ e) $\frac{1}{3}$ f) $\frac{3}{8}$

g) $\frac{7}{16}$ h) $\frac{2}{9}$ i) $\frac{7}{11}$

2. Change the following mixed numbers to decimals:

a) $2\frac{3}{4}$	b) $3\frac{3}{5}$	c) $4\frac{7}{20}$
d) $6\frac{11}{50}$	e) $5\frac{2}{3}$	f) $6\frac{7}{8}$
g) $5\frac{9}{16}$	h) $4\frac{2}{9}$	i) $5\frac{3}{7}$

Changing a decimal to a fraction

Changing a decimal to a fraction is done by knowing the 'value' of each of the digits in any decimal.

Worked examples **a)** Change 0.45 from a decimal to a fraction.

units	.	tenths	hundredths
0	.	4	5

0.45 is therefore equivalent to 4 tenths and 5 hundredths, which in turn is the same as 45 hundredths.
Therefore $0.45 = \frac{45}{100} = \frac{9}{20}$

b) Change 2.325 from a decimal to a fraction.

units	.	tenths	hundredths	thousandths
2	.	3	2	5

Therefore $2.325 = 2\frac{325}{1000} = 2\frac{13}{40}$

Exercise 4.10

1. Change the following decimals to fractions:

a) 0.5	b) 0.7	c) 0.6
d) 0.75	e) 0.825	f) 0.05
g) 0.050	h) 0.402	i) 0.0002

2. Change the following decimals to mixed numbers:

a) 2.4	b) 6.5	c) 8.2
d) 3.75	e) 10.55	f) 9.204
g) 15.455	h) 30.001	i) 1.0205

Recurring decimals

In Chapter 1 the definition of a rational number was given as any number that can be written as a fraction. These include integers, terminating decimals and recurring decimals. The examples given were:

$$0.2 = \frac{1}{5} \quad 0.3 = \frac{3}{10} \quad 7 = \frac{7}{1} \quad 1.53 = \frac{153}{100} \quad \text{and} \quad 0.\dot{2} = \frac{2}{9}$$

The first four examples are the more straightforward to understand regarding how the number can be expressed as a fraction. The fifth example shows a recurring decimal also written as a fraction. Any recurring decimal can be written as a fraction as any recurring decimal is also a rational number.

Changing a recurring decimal to a fraction

A recurring decimal is one in which the numbers after the decimal point repeat themselves infinitely. Which numbers are repeated is indicated by a dot above them.

$0.\dot{2}$ implies 0.222 222 222 222 222 …

$0.\dot{4}\dot{5}$ implies 0.454 545 454 545 …

$0.\dot{6}\dot{0}\dot{2}\dot{4}$ implies 0.602 460 246 024 …

Note, the last example is usually written as $0.\dot{6}02\dot{4}$ where a dot only appears above the first and last numbers to be repeated.

Entering $\frac{4}{9}$ into a calculator will produce 0.444 444 444…

Therefore $0.\dot{4} = \frac{4}{9}$.

The example below will prove this.

Worked examples

a) Convert $0.\dot{4}$ to a fraction.

Let $x = 0.\dot{4}$ i.e. $x = 0.444\,444\,444\,444\ldots$

$10x = 4.\dot{4}$ i.e. $10x = 4.444\,444\,444\,444\ldots$

Subtracting x from $10x$ gives:

$$\begin{array}{rcl} 10x & = & 4.444\,444\,444\,444\ldots \\ -\quad x & = & 0.444\,444\,444\,444\ldots \\ \hline 9x & = & 4 \end{array}$$

Rearranging gives $x = \frac{4}{9}$

But $x = 0.\dot{4}$

Therefore $0.\dot{4} = \frac{4}{9}$

b) Convert $0.\dot{6}\dot{8}$ to a fraction.

Let $x = 0.\dot{6}\dot{8}$ i.e. $x = 0.686\,868\,686\,868\,686\ldots$

$100x = 68.\dot{6}\dot{8}$ i.e. $100x = 68.686\,868\,686\,868\,686\ldots$

Subtracting x from $100x$ gives:

$$\begin{array}{rcl} 100x & = & 68.686\,868\,686\,868\,686\ldots \\ -\quad x & = & 0.686\,868\,686\,868\,686\ldots \\ \hline 99x & = & 68 \end{array}$$

Rearranging gives $x = \frac{68}{99}$

But $x = 0.\dot{6}\dot{8}$

Therefore $0.\dot{6}\dot{8} = \frac{68}{99}$

c) Convert $0.0\dot{3}\dot{1}$ to a fraction.

Let $x = 0.0\dot{3}\dot{1}$ i.e. $x = 0.031\ 313\ 131\ 313\ 131\ldots$

$100x = 3.\dot{1}\dot{3}$ i.e. $100x = 3.131\ 313\ 131\ 313\ 131\ldots$

Subtracting x from $100x$ gives:

$$\begin{array}{rcl} 100x & = & 3.131\ 313\ 131\ 313\ 131\ldots \\ -\quad x & = & 0.031\ 313\ 131\ 313\ 131\ldots \\ \hline 99x & = & 3.1 \end{array}$$

Multiplying both sides of the equation by 10 eliminates the decimal to give:

$990x = 31$

Rearranging gives $x = \frac{31}{990}$

But $x = 0.0\dot{3}\dot{1}$

Therefore $0.0\dot{3}\dot{1} = \frac{31}{990}$

The method is therefore to let the recurring decimal equal x and then to multiply this by a multiple of 10 so that when one is subtracted from the other either an integer (whole number) or terminating decimal (a decimal that has an end point) is left.

d) Convert $2.0\dot{4}0\dot{6}$ to a fraction.

Let $x = 2.0\dot{4}0\dot{6}$ i.e. $x = 2.040\ 640\ 640\ 640\ldots$

$1000x = 2040.\dot{6}4\dot{0}$ i.e. $1000x = 2040.640\ 640\ 640\ 640\ldots$

Subtracting x from $1000x$ gives:

$$\begin{array}{rcr} 1000x & = & 2040.640\ 640\ 640\ 640\ldots \\ -\quad x & = & 2.040\ 640\ 640\ 640\ldots \\ \hline 999x & = & 2038.6 \end{array}$$

Multiplying both sides of the equation by 10 eliminates the decimal to give:

$9990x = 20\ 386$

Rearranging gives $x = \frac{20\ 386}{9990} = 2\frac{406}{9990}$ which simplifies further to $2\frac{203}{4995}$.

But $x = 2.0\dot{4}0\dot{6}$

Therefore $2.0\dot{4}0\dot{6} = 2\frac{203}{4995}$

Exercise 4.11

1. Convert each of the following recurring decimals to fractions in their simplest form:

a) $0.\dot{3}$ b) $0.\dot{7}$

c) $0.\dot{4}\dot{2}$ d) $0.\dot{6}\dot{5}$

2. Convert each of the following recurring decimals to fractions in their simplest form:

   a) $0.0\dot{5}$ b) $0.0\dot{6}\dot{2}$
   c) $1.0\dot{2}$ d) $4.00\dot{3}\dot{8}$

3. Without using a calculator work out the sum $0.1\dot{5} + 0.0\dot{4}$ by converting each decimal to a fraction first. Give your answer as a fraction in its simplest form.

4. Without using a calculator evaluate $0.\dot{2}\dot{7} - 0.1\dot{0}\dot{6}$ by converting each decimal to a fraction first. Give your answer as a fraction in its simplest form.

Student assessment 1

1. Evaluate the following:

   a) $\frac{1}{5}$ of 60 b) $\frac{3}{5}$ of 55 c) $\frac{2}{7}$ of 21 d) $\frac{3}{4}$ of 120

2. Write the following as percentages:

   a) $\frac{3}{10}$ b) $\frac{29}{100}$ c) $\frac{1}{2}$ d) $\frac{7}{10}$
   e) $\frac{4}{5}$ f) $2\frac{19}{100}$ g) $\frac{6}{100}$ h) $\frac{3}{4}$
   i) 0.31 j) 0.07 k) 3.4 l) 2

3. Evaluate the following:

   a) $5 + 8 \times 3 - 6$ b) $15 + 45 \div 3 - 12$

4. Work out 851×27.

5. Work out $6843 \div 19$ giving your answer to 1 d.p.

6. Evaluate the following:

   a) $3\frac{3}{4} - 1\frac{11}{16}$ b) $4\frac{4}{5} \div \frac{8}{15}$

7. Change the following fractions to decimals:

   a) $\frac{2}{5}$ b) $1\frac{3}{4}$ c) $\frac{9}{11}$ d) $1\frac{2}{3}$

8. Change the following decimals to fractions. Give each fraction in its simplest form.

   a) 4.2 b) 0.06 c) 1.85 d) 2.005

9. Convert the following decimals to fractions, giving your answer in its simplest form:

   a) $0.\dot{3}\dot{7}$ b) $0.0\dot{8}\dot{0}$ c) $1.2\dot{1}$

10. Work out $0.62\dot{5} + 0.09\dot{6}$ by first converting each decimal to a fraction. Give your answer in its simplest form.

Student assessment 2

1. Evaluate the following:

a) $\frac{1}{3}$ of 63 b) $\frac{3}{8}$ of 72 c) $\frac{2}{5}$ of 55 d) $\frac{3}{13}$ of 169

2. Write the following as percentages:

a) $\frac{3}{5}$ b) $\frac{49}{100}$ c) $\frac{1}{4}$ d) $\frac{9}{10}$

e) $1\frac{1}{2}$ f) $3\frac{27}{100}$ g) $\frac{5}{100}$ h) $\frac{7}{20}$

i) 0.77 j) 0.03 k) 2.9 l) 4

3. Evaluate the following:

a) $6 \times 4 - 3 \times 8$ b) $15 \div 3 + 2 \times 7$

4. Work out 368×49.

5. Work out $7835 \div 23$ giving your answer to 1 d.p.

6. Evaluate the following:

a) $2\frac{1}{2} - \frac{4}{5}$ b) $3\frac{1}{2} \times \frac{4}{7}$

7. Change the following fractions to decimals:

a) $\frac{7}{8}$ b) $1\frac{2}{5}$ c) $\frac{8}{9}$ d) $3\frac{2}{7}$

8. Change the following decimals to fractions. Give each fraction in its simplest form.

a) 6.5 b) 0.04 c) 3.65 d) 3.008

9. Convert the following decimals to fractions, giving your answer in its simplest form:

a) $0.\dot{0}\dot{7}$ b) $0.000\dot{9}$ c) $3.0\dot{2}\dot{0}$

10. Work out $1.02\dot{5} - 0.80\dot{5}$ by first converting each decimal to a fraction. Give your answer in its simplest form.

5 Further percentages

You should already be familiar with the percentage equivalents of simple fractions and decimals as outlined in the table below.

Fraction	Decimal	Percentage
$\frac{1}{2}$	0.5	50%
$\frac{1}{4}$	0.25	25%
$\frac{3}{4}$	0.75	75%
$\frac{1}{8}$	0.125	12.5%
$\frac{3}{8}$	0.375	37.5%
$\frac{5}{8}$	0.625	62.5%
$\frac{7}{8}$	0.875	87.5%
$\frac{1}{10}$	0.1	10%
$\frac{2}{10}$ or $\frac{1}{5}$	0.2	20%
$\frac{3}{10}$	0.3	30%
$\frac{4}{10}$ or $\frac{2}{5}$	0.4	40%
$\frac{6}{10}$ or $\frac{3}{5}$	0.6	60%
$\frac{7}{10}$	0.7	70%
$\frac{8}{10}$ or $\frac{4}{5}$	0.8	80%
$\frac{9}{10}$	0.9	90%

Simple percentages

Worked examples

a) Of 100 sheep in a field, 88 are ewes.

i) What percentage of the sheep are ewes?
88 out of 100 are ewes
= 88%

ii) What percentage are not ewes?
12 out of 100
= 12%

b) Convert the following percentages into fractions and decimals:

i) 27%

$\frac{27}{100} = 0.27$

ii) 5%

$\frac{5}{100} = \frac{1}{20} = 0.05$

c) Convert $\frac{3}{16}$ to a percentage:
This example is more complicated as 16 is not a factor of 100.

Convert $\frac{3}{16}$ to a decimal first.

$3 \div 16 = 0.1875$

Convert the decimal to a percentage by multiplying by 100.

$0.1875 \times 100 = 18.75$

Therefore $\frac{3}{16} = 18.75\%$.

Exercise 5.1

1. There are 200 birds in a flock. 120 of them are female.
What percentage of the flock are:
a) female? b) male?

2. Write these fractions as percentages:
a) $\frac{7}{8}$ b) $\frac{11}{15}$ c) $\frac{7}{24}$ d) $\frac{1}{7}$

3. Convert the following percentages to decimals:
a) 39% b) 47% c) 83%
d) 7% e) 2% f) 20%

4. Convert the following decimals to percentages:
a) 0.31 b) 0.67 c) 0.09
d) 0.05 e) 0.2 f) 0.75

Calculating a percentage of a quantity

Worked examples

a) Find 25% of 300 m.

25% can be written as 0.25.
$0.25 \times 300\text{ m} = 75\text{ m}$.

b) Find 35% of 280 m.

35% can be written as 0.35.
$0.35 \times 280\text{ m} = 98\text{ m}$.

Exercise 5.2

1. Write the percentage equivalent of the following fractions:
a) $\frac{1}{4}$ b) $\frac{2}{3}$ c) $\frac{5}{8}$
d) $1\frac{4}{5}$ e) $4\frac{9}{10}$ f) $3\frac{7}{8}$

2. Write the decimal equivalent of the following:
a) $\frac{3}{4}$ b) 80% c) $\frac{1}{5}$
d) 7% e) $1\frac{7}{8}$ f) $\frac{1}{6}$

3. Evaluate the following:
a) 25% of 80 b) 80% of 125 c) 62.5% of 80
d) 30% of 120 e) 90% of 5 f) 25% of 30

4. Evaluate the following:
   a) 17% of 50 b) 50% of 17 c) 65% of 80
   d) 80% of 65 e) 7% of 250 f) 250% of 7

5. In a class of 30 students, 20% have black hair, 10% have blonde hair and 70% have brown hair. Calculate the number of students with
   a) black hair,
   b) blonde hair,
   c) brown hair.

6. A survey conducted among 120 school children looked at which type of meat they preferred. 55% said they preferred lamb, 20% said they preferred chicken, 15% preferred duck and 10% turkey. Calculate the number of children in each category.

7. A survey was carried out in a school to see what nationality its students were. Of the 220 students in the school, 65% were Australian, 20% were Pakistani, 5% were Greek and 10% belonged to other nationalities. Calculate the number of students of each nationality.

8. A shopkeeper keeps a record of the number of items he sells in one day. Of the 150 items he sold, 46% were newspapers, 24% were pens, 12% were books whilst the remaining 18% were other items. Calculate the number of each item he sold.

Expressing one quantity as a percentage of another

To express one quantity as a percentage of another, first write the first quantity as a fraction of the second and then multiply by 100.

Worked example In an examination a girl obtains 69 marks out of 75. Express this result as a percentage.

$$\frac{69}{75} \times 100\% = 92\%$$

Exercise 5.3

1. Express the first quantity as a percentage of the second.
   a) 24 out of 50 b) 46 out of 125
   c) 7 out of 20 d) 45 out of 90
   e) 9 out of 20 f) 16 out of 40
   g) 13 out of 39 h) 20 out of 35

2. A hockey team plays 42 matches. It wins 21, draws 14 and loses the rest. Express each of these results as a percentage of the total number of games played.

3. Four candidates stood in an election:

   A received 24 500 votes
   B received 18 200 votes
   C received 16 300 votes
   D received 12 000 votes

   Express each of these as a percentage of the total votes cast.

4. A car manufacturer produces 155 000 cars a year. The cars are available for sale in six different colours. The numbers sold of each colour were:

   Red 55 000
   Blue 48 000
   White 27 500
   Silver 10 200
   Green 9300
   Black 5000

   Express each of these as a percentage of the total number of cars produced. Give your answers to 1 d.p.

Percentage increases and decreases

Worked examples

a) A shop assistant has a salary of \$16 000 per month. If his salary increases by 8%, calculate:

i) the amount extra he receives a month,
ii) his new monthly salary.

i) Increase = 8% of \$16 000
= 0.08 × \$16 000 = \$1280

ii) New salary = old salary + increase
= \$16 000 + \$1280 per month
= \$17 280 per month

b) A garage increases the price of a truck by 12%. If the original price was \$14 500, calculate its new price.

The original price represents 100%, therefore the increased price can be represented as 112%.

New price = 112% of \$14 500
= 1.12 × \$14 500
= \$16 240

c) A shop is having a sale. It sells a set of tools costing \$130 at a 15% discount. Calculate the sale price of the tools.

The old price represents 100%, therefore the new price can be represented as (100 − 15)% = 85%.

85% of \$130 = 0.85 × \$130
= \$110.50

Exercise 5.4

1. Increase the following by the given percentage:

   a) 150 by 25% b) 230 by 40% c) 7000 by 2%
   d) 70 by 250% e) 80 by 12.5% f) 75 by 62%

2. Decrease the following by the given percentage:

   a) 120 by 25% b) 40 by 5% c) 90 by 90%
   d) 1000 by 10% e) 80 by 37.5% f) 75 by 42%

3. In the following questions the first number is increased to become the second number. Calculate the percentage increase in each case.

   a) $50 \rightarrow 60$ b) $75 \rightarrow 135$ c) $40 \rightarrow 84$
   d) $30 \rightarrow 31.5$ e) $18 \rightarrow 33.3$ f) $4 \rightarrow 13$

4. In the following questions the first number is decreased to become the second number. Calculate the percentage decrease in each case.

   a) $50 \rightarrow 25$ b) $80 \rightarrow 56$ c) $150 \rightarrow 142.5$
   d) $3 \rightarrow 0$ e) $550 \rightarrow 352$ f) $20 \rightarrow 19$

5. A farmer increases the yield on his farm by 15%. If his previous yield was 6500 tonnes, what is his present yield?

6. The cost of a computer in a computer store is reduced by 12.5% in a sale. If the computer was priced at $7800, what is its price in the sale?

7. A winter coat is priced at $100. In the sale its price is reduced by 25%.

   a) Calculate the sale price of the coat.
   b) After the sale its price is increased by 25% again. Calculate the coat's price after the sale.

8. A farmer takes 250 chickens to be sold at a market. In the first hour he sells 8% of his chickens. In the second hour he sells 10% of those that were left.

   a) How many chickens has he sold in total?
   b) What percentage of the original number did he manage to sell in the two hours?

9. The number of fish on a fish farm increases by approximately 10% each month. If there were originally 350 fish, calculate to the nearest 100 how many fish there would be after 12 months.

Reverse percentages

Worked examples

a) In a test Ahmed answered 92% of the questions correctly. If he answered 23 questions correctly, how many had he got wrong?

92% of the marks is equivalent to 23 questions.
1% of the marks therefore is equivalent to $\frac{23}{92}$ questions.

So 100% is equivalent to $\frac{23}{92} \times 100 = 25$ questions.

Ahmed got 2 questions wrong.

b) A boat is sold for \$15 360. This represents a profit of 28% to the seller. What did the boat originally cost the seller?

The selling price is 128% of the original cost to the seller.
128% of the original cost is \$15 360.

1% of the original cost is $\frac{\$15\,360}{128}$.

100% of the original cost is $\frac{\$15\,360}{128} \times 100$, i.e. \$12 000.

Exercise 5.5

1. Calculate the value of X in each of the following:

a) 40% of X is 240 b) 24% of X is 84
c) 85% of X is 765 d) 4% of X is 10
e) 15% of X is 18.75 f) 7% of X is 0.105

2. Calculate the value of Y in each of the following:

a) 125% of Y is 70 b) 140% of Y is 91
c) 210% of Y is 189 d) 340% of Y is 68
e) 150% of Y is 0.375 f) 144% of Y is -54.72

3. In a geography text book, 35% of the pages are coloured. If there are 98 coloured pages, how many pages are there in the whole book?

4. A town has 3500 families who own a car. If this represents 28% of the families in the town, how many families are there in total?

5. In a test Isabel scored 88%. If she got three questions incorrect, how many did she get correct?

6. Water expands when it freezes. Ice is less dense than water so it floats. If the increase in volume is 4%, what volume of water will make an iceberg of 12 700 000 m^3? Give your answer to three significant figures.

Student assessment 1

1. Find 40% of 1600 m.
2. A shop increases the price of a television set by 8%. If the present price is $320 what is the new price?
3. A car loses 55% of its value after four years. If it cost $22 500 when new, what is its value after the four years?
4. Express the first quantity as a percentage of the second.
   a) 40 cm, 2 m
   b) 25 mins, 1 hour
   c) 450 g, 2 kg
   d) 3 m, 3.5 m
   e) 70 kg, 1 tonne
   f) 75 cl, 2.5 l
5. A house is bought for $75 000 and then resold for $87 000. Calculate the percentage profit.
6. A pair of shoes is priced at $45. During a sale the price is reduced by 20%.
   a) Calculate the sale price of the shoes.
   b) What is the percentage increase in the price if after the sale it is once again restored to $45?

Student assessment 2

1. Find 30% of 2500 m.
2. In a sale a shop reduces its prices by 12.5%. What is the sale price of a desk previously costing $600?
3. In the last six years the value of a house has increased by 35%. If it cost $72 000 six years ago, what is its value now?
4. Express the first quantity as a percentage of the second.
   a) 35 mins, 2 hours
   b) 650 g, 3 kg
   c) 5 m, 4 m
   d) 15 s, 3 mins
   e) 600 kg, 3 tonnes
   f) 35 cl, 3.5 l
5. Shares in a company are bought for $600. After a year the same shares are sold for $550. Calculate the percentage depreciation.
6. In a sale the price of a jacket originally costing $1700 is reduced by $400. Any item not sold by the last day of the sale is reduced by a further 50%. If the jacket is sold on the last day of the sale:
   a) calculate the price it is finally sold for,
   b) calculate the overall percentage reduction in price.

Student assessment 3

1. Calculate the original price for each of the following:

Selling price	Profit
$3780	8%
$14 880	24%
$3.50	250%
$56.56	1%

2. Calculate the original price for each of the following:

Selling price	Loss
$350	30%
$200	20%
$8000	60%
$27 500	80%

3. In a test Ben gained 90% by answering 135 questions correctly. How many questions did he answer incorrectly?
4. A one-year-old car is worth $11 250. If its value has depreciated by 25% in that first year, calculate its price when new.
5. This year a farmer's crop yielded 50 000 tonnes. If this represents a 25% increase on last year, what was the yield last year?

Student assessment 4

1. Calculate the original price for each of the following:

Selling price	Profit
$224	12%
$62.50	150%
$660.24	26%
$38.50	285%

2. Calculate the original price for each of the following:

Selling price	Loss
$392.70	15%
$2480	38%
$3937.50	12.5%
$4675	15%

3. In an examination Sarah obtained 87.5% by gaining 105 marks. How many marks did she lose?

4. At the end of a year a factory has produced 38 500 television sets. If this represents a 10% increase in productivity on last year, calculate the number of sets that were made last year.

5. A computer manufacturer is expected to have produced 24 000 units by the end of this year. If this represents a 4% decrease on last year's output, calculate the number of units produced last year.

6. A company increased its productivity by 10% each year for the last two years. If it produced 56 265 units this year, how many units did it produce two years ago?

6 Ratio and proportion

Direct proportion

Workers in a pottery factory are paid according to how many plates they produce. The wage paid to them is said to be in **direct proportion** to the number of plates made. As the number of plates made increases so does their wage. Other workers are paid for the number of hours worked. For them the wage paid is in **direct proportion** to the number of hours worked. There are two main methods for solving problems involving direct proportion: the ratio method and the unitary method.

Worked example A bottling machine fills 500 bottles in 15 minutes. How many bottles will it fill in $1\frac{1}{2}$ hours?

Note: The time units must be the same, so for either method the $1\frac{1}{2}$ hours must be changed to 90 minutes.

The ratio method

Let x be the number of bottles filled. Then:

$$\frac{x}{90} = \frac{500}{15}$$

$$\text{so } x = \frac{500 \times 90}{15} = 3000$$

3000 bottles are filled in $1\frac{1}{2}$ hours.

The unitary method

In 15 minutes 500 bottles are filled.

Therefore in 1 minute $\frac{500}{15}$ bottles are filled.

So in 90 minutes $90 \times \frac{500}{15}$ bottles are filled.

In $1\frac{1}{2}$ hours, 3000 bottles are filled.

Exercise 6.1 Use either the ratio method or the unitary method to solve the problems below.

1. A machine prints four books in 10 minutes. How many will it print in 2 hours?
2. A farmer plants five apple trees in 25 minutes. If he continues to work at a constant rate, how long will it take him to plant 200 trees?
3. A television set uses 3 units of electricity in 2 hours. How many units will it use in 7 hours? Give your answer to the nearest unit.

4. A bricklayer lays 1500 bricks in an 8-hour day. Assuming he continues to work at the same rate, calculate:
   a) how many bricks he would expect to lay in a five-day week,
   b) how long to the nearest hour it would take him to lay 10 000 bricks.

5. A machine used to paint white lines on a road uses 250 litres of paint for each 8 km of road marked. Calculate:
   a) how many litres of paint would be needed for 200 km of road,
   b) what length of road could be marked with 4000 litres of paint.

6. An aircraft is cruising at 720 km/h and covers 1000 km. How far would it travel in the same period of time if the speed increased to 800 km/h?

7. A production line travelling at 2 m/s labels 150 tins. In the same period of time how many will it label at:
   a) 6 m/s b) 1 m/s c) 1.6 m/s?

8. A car travels at an average speed of 80 km/h for 6 hours.
   a) How far will it travel in the 6 hours?
   b) What average speed will it need to travel at in order to cover the same distance in 5 hours?

If the information is given in the form of a ratio, the method of solution is the same.

Worked example Tin and copper are mixed in the ratio 8 : 3. How much tin is needed to mix with 36 g of copper?

The ratio method

Let x grams be the mass of tin needed.

$$\frac{x}{36} = \frac{8}{3}$$

Therefore $x = \frac{8 \times 36}{3}$

$= 96$

So 96 g of tin is needed.

The unitary method

3 g of copper mixes with 8 g of tin.
1 g of copper mixes with $\frac{8}{3}$ g of tin.
So 36 g of copper mixes with $36 \times \frac{8}{3}$ g of tin.
Therefore 36 g of copper mixes with 96 g of tin.

Exercise 6.2

1. Sand and gravel are mixed in the ratio 5 : 3 to form ballast.
   a) How much gravel is mixed with 750 kg of sand?
   b) How much sand is mixed with 750 kg of gravel?

2. A recipe uses 150 g butter, 500 g flour, 50 g sugar and 100 g currants to make 18 small cakes.
   a) How much of each ingredient will be needed to make 72 cakes?
   b) How many whole cakes could be made with 1 kg of butter?

3. A paint mix uses red and white paint in a ratio of 1 : 12.
   a) How much white paint will be needed to mix with 1.4 litres of red paint?
   b) If a total of 15.5 litres of paint is mixed, calculate the amount of white paint and the amount of red paint used. Give your answers to the nearest 0.1 litre.

4. A tulip farmer sells sacks of mixed bulbs to local people. The bulbs develop into two different colours of tulips, red and yellow. The colours are packaged in a ratio of 8 : 5 respectively.
   a) If a sack contains 200 red bulbs, calculate the number of yellow bulbs.
   b) If a sack contains 351 bulbs in total, how many of each colour would you expect to find?
   c) One sack is packaged with a bulb mixture in the ratio 7 : 5 by mistake. If the sack contains 624 bulbs, how many more yellow bulbs would you expect to have compared with a normal sack of 624 bulbs?

5. A pure fruit juice is made by mixing the juices of oranges and mangoes in the ratio of 9 : 2.
   a) If 189 litres of orange juice are used, calculate the number of litres of mango juice needed.
   b) If 605 litres of the juice are made, calculate the number of litres of orange juice and mango juice used.

Divide a quantity in a given ratio

Worked examples

a) Divide 20 m in the ratio 3 : 2.

The ratio method

3 : 2 gives 5 parts.

$\frac{3}{5} \times 20\text{m} = 12\text{m}$

$\frac{2}{5} \times 20\text{m} = 8\text{m}$

20 m divided in the ratio 3 : 2 is 12 m : 8 m.

The unitary method

3 : 2 gives 5 parts.

5 parts is equivalent to 20 m.

1 part is equivalent to $\frac{20}{5}$ m.

Therefore 3 parts is $3 \times \frac{20}{5}$ m; that is 12 m.

Therefore 2 parts is $2 \times \frac{20}{5}$ m; that is 8 m.

b) A factory produces cars in red, blue, white and green in the ratio 7 : 5 : 3 : 1. Out of a production of 48 000 cars how many are white?

$7 + 5 + 3 + 1$ gives a total of 16 parts.
Therefore the total number of white cars $= \frac{3}{16} \times 48\,000 = 9000$.

Exercise 6.3

1. Divide 150 in the ratio 2 : 3.

2. Divide 72 in the ratio 2 : 3 : 4.

3. Divide 5 kg in the ratio 13 : 7.

4. Divide 45 minutes in the ratio 2 : 3.

5. Divide 1 hour in the ratio 1 : 5.

6. $\frac{7}{8}$ of a can of drink is water, the rest is syrup. What is the ratio of water to syrup?

7. $\frac{5}{9}$ of a litre carton of orange is pure orange juice, the rest is water. How many millilitres of each are in the carton?

8. 55% of students in a school are boys.
  a) What is the ratio of boys to girls?
  b) How many boys and how many girls are there if the school has 800 students?

9. A piece of wood is cut in the ratio 2 : 3. What fraction of the length is the longer piece?

10. If the piece of wood in question 9 is 80 cm long, how long is the shorter piece?

11. A gas pipe is 7 km long. A valve is positioned in such a way that it divides the length of the pipe in the ratio 4 : 3. Calculate the distance of the valve from each end of the pipe.

12. The size of the angles of a quadrilateral are in the ratio 1 : 2 : 3 : 3. Calculate the size of each angle.

13. The angles of a triangle are in the ratio 3 : 5 : 4. Calculate the size of each angle.

14. A millionaire leaves 1.4 million dollars in his will to be shared between his three children in the ratio of their ages. If they are 24, 28 and 32 years old, calculate to the nearest dollar the amount they will each receive.

15. A small company makes a profit of $8000. This is divided between the directors in the ratio of their initial investments. If Alex put $20 000 into the firm, Maria $35 000 and Ahmet $25 000, calculate the amount of the profit they will each receive.

Inverse proportion

Sometimes an increase in one quantity causes a decrease in another quantity. For example, if fruit is to be picked by hand, the more people there are picking the fruit, the less time it will take.

Worked examples

a) If 8 people can pick the apples from the trees in 6 days, how long will it take 12 people?

8 people take 6 days.
1 person will take 6×8 days.

Therefore 12 people will take $\frac{6 \times 8}{12}$ days, i.e. 4 days.

b) A cyclist averages a speed of 27 km/h for 4 hours. At what average speed would she need to cycle to cover the same distance in 3 hours?

Completing it in 1 hour would require cycling at 27×4 km/h.
Completing it in 3 hours requires cycling at

$\frac{27 \times 4}{3}$ km/h; that is 36 km/h.

Exercise 6.4

1. A teacher shares sweets among 8 students so that they get 6 each. How many sweets would they each have got had there been 12 students?

2. The table below represents the relationship between the speed and the time taken for a train to travel between two stations.

Speed (km/h)	60			120	90	50	10
Time (h)	2	3	4				

Copy and complete the table.

3. Six people can dig a trench in 8 hours.
   a) How long would it take:
      i) 4 people ii) 12 people iii) 1 person?
   b) How many people would it take to dig the trench in:
      i) 3 hours ii) 16 hours iii) 1 hour?
4. Chairs in a hall are arranged in 35 rows of 18.
   a) How many rows would there be with 21 chairs to a row?
   b) How many chairs would there be in each row if there were 15 rows?
5. A train travelling at 100 km/h takes 4 hours for a journey. How long would it take a train travelling at 60 km/h?
6. A worker in a sugar factory packs 24 cardboard boxes with 15 bags of sugar in each. If he had boxes which held 18 bags of sugar each, how many fewer boxes would be needed?
7. A swimming pool is filled in 30 hours by two identical pumps. How much quicker would it be filled if five similar pumps were used instead?

Increase and decrease by a given ratio

Worked examples

a) A photograph is 12 cm wide and 8 cm tall. It is enlarged in the ratio 3 : 2. What are the dimensions of the enlarged photograph?

3 : 2 is an enlargement of $\frac{3}{2}$. Therefore the enlarged width is 12 cm $\times \frac{3}{2}$; that is 18 cm.
The enlarged height is 8 cm $\times \frac{3}{2}$; that is 12 cm.

b) A photographic transparency 5 cm wide and 3 cm tall is projected onto a screen. If the image is 1.5 m wide:
i) calculate the ratio of the enlargement,
ii) calculate the height of the image.

i) 5 cm width is enlarged to become 150 cm.
So 1 cm width becomes $\frac{150}{5}$ cm; that is 30 cm.
Therefore the enlargement ratio is 30 : 1.
ii) The height of the image = 3 cm $\times$ 30 = 90 cm.

Exercise 6.5

1. Increase 100 by the following ratios:
   a) 8 : 5 b) 5 : 2 c) 7 : 4
   d) 11 : 10 e) 9 : 4 f) 32 : 25
2. Increase 70 by the following ratios:
   a) 4 : 3 b) 5 : 3 c) 8 : 7
   d) 9 : 4 e) 11 : 5 f) 17 : 14
3. Decrease 60 by the following ratios:
   a) 2 : 3 b) 5 : 6 c) 7 : 12
   d) 3 : 5 e) 1 : 4 f) 13 : 15

4. Decrease 30 by the following ratios:
   a) 3 : 4 b) 2 : 9 c) 7 : 12
   d) 3 : 16 e) 5 : 8 f) 9 : 20
5. Increase 40 by a ratio of 5 : 4.
6. Decrease 40 by a ratio of 4 : 5.
7. Increase 150 by a ratio of 7 : 5.
8. Decrease 210 by a ratio of 3 : 7.

Exercise 6.6

1. A photograph measuring 8 cm by 6 cm is enlarged by a ratio of 11 : 4. What are the dimensions of the new print?
2. A photocopier enlarges in the ratio 7 : 4. What would be the new size of a diagram measuring 16 cm by 12 cm?
3. A drawing measuring 10 cm by 16 cm needs to be enlarged. The dimensions of the enlargement need to be 25 cm by 40 cm. Calculate the enlargement needed and express it as a ratio.
4. A banner needs to be enlarged from its original format. The dimensions of the original are 4 cm tall by 25 cm wide. The enlarged banner needs to be at least 8 m wide but no more than 1.4 m tall. Calculate the minimum and maximum ratios of enlargement possible.
5. A rectangle measuring 7 cm by 4 cm is enlarged by a ratio of 2 : 1.
   a) What is the area of:
      i) the original rectangle?
      ii) the enlarged rectangle?
   b) By what ratio has the area been enlarged?
6. A square of side length 3 cm is enlarged by a ratio of 3 : 1.
   a) What is the area of:
      i) the original square?
      ii) the enlarged square?
   b) By what ratio has the area been enlarged?
7. A cuboid measuring 3 cm by 5 cm by 2 cm is enlarged by a ratio of 2 : 1.
   a) What is the volume of:
      i) the original cuboid?
      ii) the enlarged cuboid?
   b) By what ratio has the volume been increased?
8. A cube of side 4 cm is enlarged by a ratio of 3 : 1.
   a) What is the volume of:
      i) the original cube?
      ii) the enlarged cube?
   b) By what ratio has the volume been increased?

9. The triangle is to be reduced by a ratio of 1 : 2.

a) Calculate the area of the original triangle.
b) Calculate the area of the reduced triangle.
c) Calculate the ratio by which the area of the triangle has been reduced.

10. From questions 5–9 can you conclude what happens to two- and three-dimensional figures when they are either enlarged or reduced?

Student assessment 1

1. A boat travels at an average speed of 15 km/h for 1 hour.
a) Calculate the distance it travels in one hour.
b) What average speed will the boat need to travel at in order to cover the same distance in $2\frac{1}{2}$ hours?

2. A ruler 30 cm long is broken into two parts in the ratio 8 : 7. How long are the two parts?

3. A recipe needs 400 g of flour to make 8 cakes. How much flour would be needed in order to make 24 cakes?

4. To make 6 jam tarts, 120 g of jam is needed. How much jam is needed to make 10 tarts?

5. The scale of a map is 1 : 25 000.
a) Two villages are 8 cm apart on the map. How far apart are they in real life? Give your answer in kilometres.
b) The distance from a village to the edge of a lake is 12 km in real life. How far apart would they be on the map? Give your answer in centimetres.

6. A motorbike uses petrol and oil mixed in the ratio 13 : 2.
a) How much of each is there in 30 litres of mixture?
b) How much petrol would be mixed with 500 ml of oil?

7. a) A model car is a $\frac{1}{40}$ scale model. Express this as a ratio.
b) If the length of the real car is 5.5 m, what is the length of the model car?

8. An aunt gives a brother and sister $2000 to be divided in the ratio of their ages. If the girl is 13 years old and the boy 12 years old, how much will each get?

9. The angles of a triangle are in the ratio 2 : 5 : 8. Find the size of each of the angles.

10. A photocopying machine is capable of making 50 copies each minute.
    a) If four identical copiers are used simultaneously how long would it take to make a total of 50 copies?
    b) How many copiers would be needed to make 6000 copies in 15 minutes?

11. It takes 16 hours for three bricklayers to build a wall. Calculate how long it would take for eight bricklayers to build a similar wall.

12. A photocopier enlarges by a ratio of 7 : 4. A picture measures 6 cm by 4 cm. How many consecutive enlargements can be made so that the largest possible picture will fit on a sheet measuring 30 cm by 20 cm?

Student assessment 2

1. A cyclist travels at an average speed of 20 km/h for 1.5 hours.
   a) Calculate the distance she travels in 1.5 hours.
   b) What average speed will the cyclist need to travel in order to cover the same distance in 1 hour?

2. A piece of wood is cut in the ratio 3 : 7.
   a) What fraction of the whole is the longer piece?
   b) If the wood is 1.5 m long, how long is the shorter piece?

3. A recipe for two people requires $\frac{1}{4}$ kg of rice to 150 g of meat.
   a) How much meat would be needed for five people?
   b) How much rice would there be in 1 kg of the final dish?

4. The scale of a map is 1 : 10 000.
   a) Two rivers are 4.5 cm apart on the map, how far apart are they in real life? Give your answer in metres.
   b) Two towns are 8 km apart in real life. How far apart are they on the map? Give your answer in centimetres.

5. a) A model train is a $\frac{1}{25}$ scale model. Express this as a ratio.
   b) If the length of the model engine is 7 cm, what is the true length of the engine?

6. Divide 3 tonnes in the ratio 2 : 5 : 13.

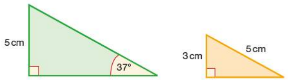
7. The ratio of the angles of a quadrilateral is 2 : 3 : 3 : 4. Calculate the size of each of the angles.

8. The ratio of the interior angles of a pentagon is 2 : 3 : 4 : 4 : 5. Calculate the size of the largest angle.

NB: All diagrams are not drawn to scale.

9. A large swimming pool takes 36 hours to fill using three identical pumps.
 a) How long would it take to fill using eight identical pumps?
 b) If the pool needs to be filled in 9 hours, how many pumps will be needed?

10. The first triangle is an enlargement of the second. Calculate the size of the missing sides and angles.



11. A tap issuing water at a rate of 1.2 litres per minute fills a container in 4 minutes.
 a) How long would it take to fill the same container if the rate was decreased to 1 litre per minute? Give your answer in minutes and seconds.
 b) If the container is to be filled in 3 minutes, calculate the rate at which the water should flow.

12. A map measuring 60 cm by 25 cm is reduced twice in the ratio 3 : 5. Calculate the final dimensions of the map.

7 Indices and standard form

The index refers to the power to which a number is raised. In the example 5^3 the number 5 is raised to the power 3. The 3 is known as the **index**. Indices is the plural of index.

Worked examples

a) $5^3 = 5 \times 5 \times 5$
$= 125$

b) $7^4 = 7 \times 7 \times 7 \times 7$
$= 2401$

c) $3^1 = 3$

Laws of indices

When working with numbers involving indices there are three basic laws which can be applied. These are:

(1) $a^m \times a^n = a^{m+n}$

(2) $a^m \div a^n$ or $\frac{a^m}{a^n} = a^{m-n}$

(3) $(a^m)^n = a^{mn}$

Positive indices

Worked examples

a) Simplify $4^3 \times 4^2$.

$4^3 \times 4^2 = 4^{(3+2)}$
$= 4^5$

b) Simplify $2^5 \div 2^3$.

$2^5 \div 2^3 = 2^{(5-3)}$
$= 2^2$

c) Evaluate $3^3 \times 3^4$.

$3^3 \times 3^4 = 3^{(3+4)}$
$= 3^7$
$= 2187$

d) Evaluate $(4^2)^3$.

$(4^2)^3 = 4^{(2\times3)}$
$= 4^6$
$= 4096$

Exercise 7.1

1. Using indices, simplify the following expressions:

a) $3 \times 3 \times 3$
b) $2 \times 2 \times 2 \times 2 \times 2$
c) 4×4
d) $6 \times 6 \times 6 \times 6$
e) $8 \times 8 \times 8 \times 8 \times 8 \times 8$
f) 5

2. Simplify the following using indices:

a) $2 \times 2 \times 2 \times 3 \times 3$
b) $4 \times 4 \times 4 \times 4 \times 4 \times 5 \times 5$
c) $3 \times 3 \times 4 \times 4 \times 4 \times 5 \times 5$
d) $2 \times 7 \times 7 \times 7 \times 7$
e) $1 \times 1 \times 6 \times 6$
f) $3 \times 3 \times 3 \times 4 \times 4 \times 6 \times 6 \times 6 \times 6 \times 6$

3. Write out the following in full:

a) 4^2 b) 5^7
c) 3^5 d) $4^3 \times 6^3$
e) $7^2 \times 2^7$ f) $3^2 \times 4^3 \times 2^4$

4. Without a calculator work out the value of the following:

a) 2^5 b) 3^4
c) 8^2 d) 6^3
e) 10^6 f) 4^4
g) $2^3 \times 3^2$ h) $10^3 \times 5^3$

Exercise 7.2

1. Simplify the following using indices:

a) $3^2 \times 3^4$ b) $8^5 \times 8^2$
c) $5^2 \times 5^4 \times 5^3$ d) $4^3 \times 4^5 \times 4^2$
e) $2^1 \times 2^3$ f) $6^2 \times 3^2 \times 3^3 \times 6^4$
g) $4^5 \times 4^3 \times 5^5 \times 5^4 \times 6^2$ h) $2^4 \times 5^7 \times 5^3 \times 6^2 \times 6^6$

2. Simplify the following:

a) $4^6 \div 4^2$ b) $5^7 \div 5^4$
c) $2^5 \div 2^4$ d) $6^5 \div 6^2$
e) $\frac{6^5}{6^2}$ f) $\frac{8^6}{8^5}$
g) $\frac{4^8}{4^5}$ h) $\frac{3^9}{3^2}$

3. Simplify the following:

a) $(5^2)^2$ b) $(4^3)^4$
c) $(10^2)^5$ d) $(3^3)^5$
e) $(6^2)^4$ f) $(8^2)^3$

4. Simplify the following:

a) $\frac{2^2 \times 2^4}{2^3}$ b) $\frac{3^4 \times 3^2}{3^5}$
c) $\frac{5^6 \times 5^7}{5^2 \times 5^8}$ d) $\frac{(4^2)^5 \times 4^2}{4^7}$
e) $\frac{4^4 \times 2^5 \times 4^2}{4^3 \times 2^3}$ f) $\frac{6^3 \times 6^3 \times 8^5 \times 8^6}{8^6 \times 6^2}$
g) $\frac{(5^5)^2 \times (4^4)^3}{5^8 \times 4^9}$ h) $\frac{(6^3)^4 \times 6^3 \times 4^9}{6^8 \times 4^8}$

The zero index

The zero index indicates that a number is raised to the power 0. A number raised to the power 0 is equal to 1. This can be explained by applying the laws of indices.

$$a^m \div a^n = a^{m-n} \quad \text{therefore} \quad \frac{a^m}{a^m} = a^{m-m}$$

$$= a^0$$

However, $$\frac{a^m}{a^m} = 1$$

therefore $$a^0 = 1$$

Negative indices

A negative index indicates that a number is being raised to a negative power: e.g. 4^{-3}.

Another law of indices states that $a^{-m} = \frac{1}{a^m}$. This can be proved as follows.

$$a^{-m} = a^{0-m}$$

$$= \frac{a^0}{a^m} \text{ (from the second law of indices)}$$

$$= \frac{1}{a^m}$$

therefore $a^{-m} = \frac{1}{a^m}$

Exercise 7.3 Without using a calculator, evaluate the following:

1. a) $2^3 \times 2^0$ b) $5^2 \div 6^0$
c) $5^2 \times 5^{-2}$ d) $6^3 \times 6^{-3}$
e) $(4^0)^2$ f) $4^0 \div 2^2$

2. a) 4^{-1} b) 3^{-2}
c) 6×10^{-2} d) 5×10^{-3}
e) 100×10^{-2} f) 10^{-3}

3. a) 9×3^{-2} b) 16×2^{-3}
c) 64×2^{-4} d) 4×2^{-3}
e) 36×6^{-3} f) 100×10^{-1}

4. a) $\frac{3}{2^{-2}}$ b) $\frac{4}{2^{-3}}$
c) $\frac{9}{5^{-2}}$ d) $\frac{5}{4^{-2}}$
e) $\frac{7^{-3}}{7^{-4}}$ f) $\frac{8^{-6}}{8^{-8}}$

Exponential equations

Equations that involve indices as unknowns are known as **exponential equations**.

Worked examples

a) Find the value of x if $2^x = 32$.

32 can be expressed as a power of 2,
$32 = 2^5$.

Therefore $2^x = 2^5$

$x = 5$

b) Find the value of m if $3^{(m-1)} = 81$.

81 can be expressed as a power of 3,
$81 = 3^4$.

Therefore $3^{(m-1)} = 3^4$

$m - 1 = 4$

$m = 5$

Exercise 7.4

1. Find the value of x in each of the following:

a) $2^x = 4$ b) $2^x = 16$
c) $4^x = 64$ d) $10^x = 1000$
e) $5^x = 625$ f) $3^x = 1$

2. Find the value of z in each of the following:

a) $2^{(z-1)} = 8$ b) $3^{(z+2)} = 27$
c) $4^{2z} = 64$ d) $10^{(z+1)} = 1$
e) $3^z = 9^{(z-1)}$ f) $5^z = 125^z$

3. Find the value of n in each of the following:

a) $(\frac{1}{2})^n = 8$ b) $(\frac{1}{3})^n = 81$
c) $(\frac{1}{2})^n = 32$ d) $(\frac{1}{2})^n = 4^{(n+1)}$
e) $(\frac{1}{2})^{(n+1)} = 2$ f) $(\frac{1}{16})^n = 4$

4. Find the value of x in each of the following:

a) $3^{-x} = 27$ b) $2^{-x} = 128$
c) $2^{(-x+3)} = 64$ d) $4^{-x} = \frac{1}{16}$
e) $2^{-x} = \frac{1}{256}$ f) $3^{(-x+1)} = \frac{1}{81}$

Standard form

Standard form is also known as standard index form or sometimes as scientific notation. It involves writing large numbers or very small numbers in terms of powers of 10.

Positive indices and large numbers

$$100 = 1 \times 10^2$$
$$1000 = 1 \times 10^3$$
$$10\,000 = 1 \times 10^4$$
$$3000 = 3 \times 10^3$$

For a number to be in standard form it must take the form $A \times 10^n$ where the index n is a positive or negative integer and A must lie in the range $1 \leqslant A < 10$.

e.g. 3100 can be written in many different ways:

3.1×10^3 $\quad 31 \times 10^2$ $\quad 0.31 \times 10^4$ $\quad$ etc.

However, only 3.1×10^3 satisfies the above conditions and therefore is the only one which is written in standard form.

Worked examples

a) Write 72 000 in standard form.

7.2×10^4

b) Write 4×10^4 as an ordinary number.

$4 \times 10^4 = 4 \times 10\,000$
$= 40\,000$

c) Multiply the following and write your answer in standard form:

600×4000
$= 2\,400\,000$
$= 2.4 \times 10^6$

d) Multiply the following and write your answer in standard form:

$(2.4 \times 10^4) \times (5 \times 10^7)$
$= 12 \times 10^{11}$
$= 1.2 \times 10^{12}$ when written in standard form

e) Divide the following and write your answer in standard form:

$(6.4 \times 10^7) \div (1.6 \times 10^3)$
$= 4 \times 10^4$

f) Add the following and write your answer in standard form:

$(3.8 \times 10^6) + (8.7 \times 10^4)$

Changing the indices to the same value gives the sum:

$(380 \times 10^4) + (8.7 \times 10^4)$
$= 388.7 \times 10^4$
$= 3.887 \times 10^6$ when written in standard form

g) Subtract the following and write your answer in standard form:

$(6.5 \times 10^7) - (9.2 \times 10^5)$

Changing the indices to the same value gives

$(650 \times 10^5) - (9.2 \times 10^5)$
$= 640.8 \times 10^5$
$= 6.408 \times 10^7$ when written in standard form

Exercise 7.5

1. Which of the following are not in standard form?
a) 6.2×10^5 b) 7.834×10^{16}
c) 8.0×10^5 d) 0.46×10^7
e) 82.3×10^6 f) 6.75×10^1

2. Write the following numbers in standard form:
a) 600 000 b) 48 000 000
c) 784 000 000 000 d) 534 000
e) 7 million f) 8.5 million

3. Write the following in standard form:
a) 68×10^5 b) 720×10^6
c) 8×10^5 d) 0.75×10^8
e) 0.4×10^{10} f) 50×10^6

4. Write the following as ordinary numbers:
a) 3.8×10^3 b) 4.25×10^6
c) 9.003×10^7 d) 1.01×10^5

5. Multiply the following and write your answers in standard form:
a) 200×3000 b) 6000×4000
c) 7 million $\times$ 20 d) 500 $\times$ 6 million
e) 3 million $\times$ 4 million f) 4500×4000

6. Light from the Sun takes approximately 8 minutes to reach Earth. If light travels at a speed of 3×10^8 m/s, calculate to three significant figures (s.f.) the distance from the Sun to the Earth.

7. Find the value of the following and write your answers in standard form:
a) $(4.4 \times 10^3) \times (2 \times 10^5)$ b) $(6.8 \times 10^7) \times (3 \times 10^3)$
c) $(4 \times 10^5) \times (8.3 \times 10^5)$ d) $(5 \times 10^9) \times (8.4 \times 10^{12})$
e) $(8.5 \times 10^6) \times (6 \times 10^{15})$ f) $(5.0 \times 10^{12})^2$

8. Find the value of the following and write your answers in standard form:
a) $(3.8 \times 10^8) \div (1.9 \times 10^6)$ b) $(6.75 \times 10^9) \div (2.25 \times 10^4)$
c) $(9.6 \times 10^{11}) \div (2.4 \times 10^5)$ d) $(1.8 \times 10^{12}) \div (9.0 \times 10^7)$
e) $(2.3 \times 10^{11}) \div (9.2 \times 10^4)$ f) $(2.4 \times 10^8) \div (6.0 \times 10^3)$

9. Find the value of the following and write your answers in standard form:
   a) $(3.8 \times 10^5) + (4.6 \times 10^4)$ b) $(7.9 \times 10^9) + (5.8 \times 10^8)$
   c) $(6.3 \times 10^7) + (8.8 \times 10^5)$ d) $(3.15 \times 10^9) + (7.0 \times 10^6)$
   e) $(5.3 \times 10^8) - (8.0 \times 10^7)$ f) $(6.5 \times 10^7) - (4.9 \times 10^6)$
   g) $(8.93 \times 10^{10}) - (7.8 \times 10^9)$ h) $(4.07 \times 10^7) - (5.1 \times 10^6)$

Negative indices and small numbers

A negative index is used when writing a number between 0 and 1 in standard form.

e.g.
$$
\begin{aligned}
100 &= 1 \times 10^2 \\
10 &= 1 \times 10^1 \\
1 &= 1 \times 10^0 \\
0.1 &= 1 \times 10^{-1} \\
0.01 &= 1 \times 10^{-2} \\
0.001 &= 1 \times 10^{-3} \\
0.0001 &= 1 \times 10^{-4}
\end{aligned}
$$

Note that A must still lie within the range $1 \leqslant A < 10$.

Worked examples

a) Write 0.0032 in standard form.

3.2×10^{-3}

b) Write 1.8×10^{-4} as an ordinary number.

$$
\begin{aligned}
1.8 \times 10^{-4} &= 1.8 \div 10^4 \\
&= 1.8 \div 10\,000 \\
&= 0.000\,18
\end{aligned}
$$

c) Write the following numbers in order of magnitude, starting with the largest:

3.6×10^{-3} 5.2×10^{-5} 1×10^{-2} 8.35×10^{-2} 6.08×10^{-8}

8.35×10^{-2} 1×10^{-2} 3.6×10^{-3} 5.2×10^{-5} 6.08×10^{-8}

Exercise 7.6

1. Write the following numbers in standard form:
   a) 0.0006 b) 0.000 053
   c) 0.000 864 d) 0.000 000 088
   e) 0.000 000 7 f) 0.000 414 5

2. Write the following numbers in standard form:
   a) 68×10^{-5} b) 750×10^{-9}
   c) 42×10^{-11} d) 0.08×10^{-7}
   e) 0.057×10^{-9} f) 0.4×10^{-10}

3. Write the following as ordinary numbers:
   a) 8×10^{-3} b) 4.2×10^{-4}
   c) 9.03×10^{-2} d) 1.01×10^{-5}

4. Deduce the value of n in each of the following cases:

a) $0.000\,25 = 2.5 \times 10^n$ b) $0.003\,57 = 3.57 \times 10^n$

c) $0.000\,000\,06 = 6 \times 10^n$ d) $0.004^2 = 1.6 \times 10^n$

e) $0.000\,65^2 = 4.225 \times 10^n$ f) $0.0002^n = 8 \times 10^{-12}$

5. Write these numbers in order of magnitude, starting with the largest:

3.2×10^{-4} 6.8×10^5 5.57×10^{-9} 6.2×10^3
5.8×10^{-7} 6.741×10^{-4} 8.414×10^2

● Fractional indices

$16^{\frac{1}{2}}$ can be written as $(4^2)^{\frac{1}{2}}$.

$$(4^2)^{\frac{1}{2}} = 4^{(2 \times \frac{1}{2})}$$
$$= 4^1$$
$$= 4$$

Therefore $16^{\frac{1}{2}} = 4$

but $\sqrt{16} = 4$

therefore $16^{\frac{1}{2}} = \sqrt{16}$

Similarly:

$27^{\frac{1}{3}}$ can be written as $(3^3)^{\frac{1}{3}}$

$$(3^3)^{\frac{1}{3}} = 3^{(3 \times \frac{1}{3})}$$
$$= 3^1$$
$$= 3$$

Therefore $27^{\frac{1}{3}} = 3$

but $\sqrt[3]{27} = 3$

therefore $27^{\frac{1}{3}} = \sqrt[3]{27}$

In general:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$a^{\frac{m}{n}} = \sqrt[n]{(a^m)} \text{ or } (\sqrt[n]{a})^m$$

Worked examples **a)** Evaluate $16^{\frac{1}{4}}$ without the use of a calculator.

$16^{\frac{1}{4}} = \sqrt[4]{16}$ Alternatively: $16^{\frac{1}{4}} = (2^4)^{\frac{1}{4}}$

$= \sqrt[4]{(2^4)}$ $= 2^1$
$= 2$ $= 2$

b) Evaluate $25^{\frac{3}{2}}$ without the use of a calculator.

$$
\begin{aligned}
25^{\frac{3}{2}} &= (25^{\frac{1}{2}})^3 \\
&= (\sqrt{25})^3 \\
&= 5^3 \\
&= 125
\end{aligned}
\qquad \text{Alternatively:} \quad
\begin{aligned}
25^{\frac{3}{2}} &= (5^2)^{\frac{3}{2}} \\
&= 5^3 \\
&= 125
\end{aligned}
$$

c) Solve $32^x = 2$

32 is 2^5 so $\sqrt[5]{32} = 2$
or $32^{\frac{1}{5}} = 2$
therefore $x = \frac{1}{5}$

d) Solve $125^x = 5$

125 is 5^3 so $\sqrt[3]{125} = 5$
or $125^{\frac{1}{3}} = 5$
therefore $x = \frac{1}{3}$

Exercise 7.7

Evaluate the following without the use of a calculator:

1. a) $16^{\frac{1}{2}}$ b) $25^{\frac{1}{2}}$ c) $100^{\frac{1}{2}}$
d) $27^{\frac{1}{3}}$ e) $81^{\frac{1}{2}}$ f) $1000^{\frac{1}{3}}$

2. a) $16^{\frac{1}{4}}$ b) $81^{\frac{1}{4}}$ c) $32^{\frac{1}{5}}$
d) $64^{\frac{1}{6}}$ e) $216^{\frac{1}{3}}$ f) $256^{\frac{1}{4}}$

3. a) $4^{\frac{3}{2}}$ b) $4^{\frac{5}{2}}$ c) $9^{\frac{3}{2}}$
d) $16^{\frac{3}{2}}$ e) $1^{\frac{5}{2}}$ f) $27^{\frac{2}{3}}$

4. a) $125^{\frac{2}{3}}$ b) $32^{\frac{3}{5}}$ c) $64^{\frac{5}{6}}$
d) $1000^{\frac{2}{3}}$ e) $16^{\frac{5}{4}}$ f) $81^{\frac{3}{4}}$

5. a) solve $16^x = 4$ b) solve $8^x = 2$
c) solve $9^x = 3$ d) solve $27^x = 3$
e) solve $100^x = 10$ f) solve $64^x = 2$

6. a) solve $1000^x = 10$ b) solve $49^x = 7$
c) solve $81^x = 3$ d) solve $343^x = 7$
e) solve $1\,000\,000^x = 10$ f) solve $216^x = 6$

Exercise 7.8

Evaluate the following without the use of a calculator:

1. a) $\dfrac{27^{\frac{2}{3}}}{3^2}$ b) $\dfrac{7^{\frac{3}{2}}}{\sqrt{7}}$ c) $\dfrac{4^{\frac{5}{2}}}{4^2}$

d) $\dfrac{16^{\frac{3}{2}}}{2^6}$ e) $\dfrac{27^{\frac{5}{3}}}{\sqrt{9}}$ f) $\dfrac{6^{\frac{4}{3}}}{6^{\frac{1}{3}}}$

2. a) $5^{\frac{2}{3}} \times 5^{\frac{4}{3}}$ b) $4^{\frac{1}{4}} \times 4^{\frac{1}{4}}$ c) 8×2^{-2}
d) $3^{\frac{4}{3}} \times 3^{\frac{5}{3}}$ e) $2^{-2} \times 16$ f) $8^{\frac{5}{3}} \times 8^{-\frac{4}{3}}$

3. a) $\dfrac{2^{\frac{1}{2}} \times 2^{\frac{5}{2}}}{2}$ b) $\dfrac{4^{\frac{5}{6}} \times 4^{\frac{1}{6}}}{4^{\frac{1}{2}}}$ c) $\dfrac{2^3 \times 8^{\frac{3}{2}}}{\sqrt{8}}$

   d) $\dfrac{(3^2)^{\frac{3}{2}} \times 3^{-\frac{1}{2}}}{3^{\frac{1}{2}}}$ e) $\dfrac{8^{\frac{1}{3}} + 7}{27^{\frac{1}{3}}}$ f) $\dfrac{9^{\frac{1}{2}} \times 3^{\frac{5}{2}}}{3^{\frac{2}{3}} \times 3^{-\frac{1}{6}}}$

Student assessment 1

1. Using indices, simplify the following:
   a) $2 \times 2 \times 2 \times 5 \times 5$ b) $2 \times 2 \times 3 \times 3 \times 3 \times 3 \times 3$
2. Write the following out in full:
   a) 4^3 b) 6^4
3. Work out the value of the following without using a calculator:
   a) $2^3 \times 10^2$ b) $1^4 \times 3^3$
4. Simplify the following using indices:
   a) $3^4 \times 3^3$ b) $6^3 \times 6^2 \times 3^4 \times 3^5$
   c) $\dfrac{4^5}{2^3}$ d) $\dfrac{(6^2)^3}{6^5}$
   e) $\dfrac{3^5 \times 4^2}{3^3 \times 4^0}$ f) $\dfrac{4^{-2} \times 2^6}{2^2}$
5. Without using a calculator, evaluate the following:
   a) $2^4 \times 2^{-2}$ b) $\dfrac{3^5}{3^3}$
   c) $\dfrac{5^{-5}}{5^{-6}}$ d) $\dfrac{2^5 \times 4^{-3}}{2^{-1}}$
6. Find the value of x in each of the following:
   a) $2^{(x-2)} = 32$ b) $\dfrac{1}{4^x} = 16$
   c) $5^{(-x+2)} = 125$ d) $8^{-x} = \frac{1}{2}$

Student assessment 2

1. Using indices, simplify the following:
   a) $3 \times 2 \times 2 \times 3 \times 27$
   b) $2 \times 2 \times 4 \times 4 \times 4 \times 2 \times 32$
2. Write the following out in full:
   a) 6^5 b) 2^{-5}
3. Work out the value of the following without using a calculator:
   a) $3^3 \times 10^3$ b) $1^{-4} \times 5^3$

4. Simplify the following using indices:

a) $2^4 \times 2^3$ b) $7^5 \times 7^2 \times 3^4 \times 3^8$

c) $\frac{4^8}{2^{10}}$ d) $\frac{(3^3)^4}{27^3}$

e) $\frac{7^6 \times 4^2}{4^3 \times 7^6}$ f) $\frac{8^{-2} \times 2^6}{2^{-2}}$

5. Without using a calculator, evaluate the following:

a) $5^2 \times 5^{-1}$ b) $\frac{4^5}{4^3}$

c) $\frac{7^{-5}}{7^{-7}}$ d) $\frac{3^{-5} \times 4^2}{3^{-6}}$

6. Find the value of x in each of the following:

a) $2^{(2x+2)} = 128$ b) $\frac{1}{4^{-x}} = \frac{1}{2}$

c) $3^{(-x+4)} = 81$ d) $8^{-3x} = \frac{1}{4}$

Student assessment 3

1. Write the following numbers in standard form:

a) 8 million b) 0.000 72

c) 75 000 000 000 d) 0.0004

e) 4.75 billion f) 0.000 000 64

2. Write the following as ordinary numbers:

a) 2.07×10^4 b) 1.45×10^{-3}

c) 5.23×10^{-2}

3. Write the following numbers in order of magnitude, starting with the smallest:

6.2×10^7 5.5×10^{-3} 4.21×10^7 4.9×10^8 3.6×10^{-5} 7.41×10^{-9}

4. Write the following numbers:

a) in standard form,

b) in order of magnitude, starting with the largest.

6 million 820 000 0.0044 0.8 52 000

5. Deduce the value of n in each of the following:

a) $620 = 6.2 \times 10^n$ b) $555\,000\,000 = 5.55 \times 10^n$

c) $0.000\,45 = 4.5 \times 10^n$ d) $500^2 = 2.5 \times 10^n$

e) $0.0035^2 = 1.225 \times 10^n$ f) $0.04^3 = 6.4 \times 10^n$

6. Write the answers to the following calculations in standard form:

a) $4000 \times 30\,000$ b) $(2.8 \times 10^5) \times (2.0 \times 10^3)$

c) $(3.2 \times 10^9) \div (1.6 \times 10^4)$ d) $(2.4 \times 10^8) \div (9.6 \times 10^2)$

7. The speed of light is 3×10^8 m/s. Venus is 108 million km from the Sun. Calculate the number of minutes it takes for sunlight to reach Venus.

8. A star system is 500 light years away from Earth. The speed of light is 3×10^5 km/s. Calculate the distance the star system is from Earth. Give your answer in kilometres and written in standard form.

Student assessment 4

1. Write the following numbers in standard form:
   a) 6 million b) 0.0045
   c) 3 800 000 000 d) 0.000 000 361
   e) 460 million f) 3

2. Write the following as ordinary numbers:
   a) 8.112×10^6 b) 4.4×10^5
   c) 3.05×10^{-4}

3. Write the following numbers in order of magnitude, starting with the largest:

   3.6×10^2 2.1×10^{-3} 9×10^1 4.05×10^8 1.5×10^{-2}
   7.2×10^{-3}

4. Write the following numbers:
   a) in standard form,
   b) in order of magnitude, starting with the smallest.

   15 million 430 000 0.000 435 4.8 0.0085

5. Deduce the value of n in each of the following:
   a) $4750 = 4.75 \times 10^n$ b) $6\,440\,000\,000 = 6.44 \times 10^n$
   c) $0.0040 = 4.0 \times 10^n$ d) $1000^2 = 1 \times 10^n$
   e) $0.9^3 = 7.29 \times 10^n$ f) $800^3 = 5.12 \times 10^n$

6. Write the answers to the following calculations in standard form:
   a) $50\,000 \times 2400$ b) $(3.7 \times 10^6) \times (4.0 \times 10^4)$
   c) $(5.8 \times 10^7) + (9.3 \times 10^6)$ d) $(4.7 \times 10^6) - (8.2 \times 10^5)$

7. The speed of light is 3×10^8 m/s. Jupiter is 778 million km from the Sun. Calculate the number of minutes it takes for sunlight to reach Jupiter.

8. A star is 300 light years away from Earth. The speed of light is 3×10^5 km/s. Calculate the distance from the star to Earth. Give your answer in kilometres and written in standard form.

Student assessment 5

1. Evaluate the following without the use of a calculator:

   a) $81^{\frac{1}{2}}$ b) $27^{\frac{1}{3}}$ c) $9^{\frac{1}{2}}$ d) $625^{\frac{3}{4}}$

   e) $343^{\frac{2}{3}}$ f) $16^{-\frac{1}{4}}$ g) $\dfrac{1}{25^{-\frac{1}{2}}}$ h) $\dfrac{2}{16^{-\frac{3}{4}}}$

2. Evaluate the following without the use of a calculator:

   a) $\dfrac{16^{\frac{1}{2}}}{2^2}$ b) $\dfrac{9^{\frac{5}{2}}}{3^3}$

   c) $\dfrac{8^{\frac{4}{3}}}{8^{\frac{2}{3}}}$ d) $5^{\frac{6}{5}} \times 5^{\frac{4}{5}}$

   e) $4^{\frac{3}{2}} \times 2^{-2}$ f) $\dfrac{27^{\frac{2}{3}} \times 3^{-2}}{4^{-\frac{3}{2}}}$

   g) $\dfrac{(4^3)^{-\frac{1}{2}} \times 2^{\frac{3}{2}}}{2^{-\frac{3}{2}}}$ h) $\dfrac{(5^{\frac{2}{3}})^{\frac{1}{2}} \times 5^{\frac{2}{3}}}{3^{-2}}$

3. Draw a pair of axes with x from -4 to 4 and y from 0 to 10.

   a) Plot a graph of $y = 3^{\frac{x}{2}}$.

   b) Use your graph to estimate when $3^{\frac{x}{2}} = 5$.

Student assessment 6

1. Evaluate the following without the use of a calculator:

   a) $64^{\frac{1}{6}}$ b) $27^{\frac{4}{3}}$ c) $9^{-\frac{1}{2}}$ d) $512^{\frac{2}{3}}$

   e) $\sqrt[3]{27}$ f) $\sqrt[4]{16}$ g) $\dfrac{1}{36^{-\frac{1}{2}}}$ h) $\dfrac{2}{64^{-\frac{2}{3}}}$

2. Evaluate the following without the use of a calculator:

   a) $\dfrac{25^{\frac{1}{2}}}{9^{-\frac{1}{2}}}$ b) $\dfrac{4^{\frac{5}{2}}}{2^3}$

   c) $\dfrac{27^{\frac{4}{3}}}{3^3}$ d) $25^{\frac{3}{2}} \times 5^2$

   e) $4^{\frac{6}{4}} \times 4^{-\frac{1}{2}}$ f) $\dfrac{27^{\frac{2}{3}} \times 3^{-3}}{9^{-\frac{1}{2}}}$

   g) $\dfrac{(4^2)^{-\frac{1}{4}} \times 9^{\frac{3}{2}}}{\left(\frac{1}{4}\right)^{\frac{1}{2}}}$ h) $\dfrac{(5^{\frac{1}{3}})^{\frac{1}{2}} \times 5^{\frac{5}{6}}}{4^{-\frac{1}{2}}}$

3. Draw a pair of axes with x from -4 to 4 and y from 0 to 18.

   a) Plot a graph of $y = 4^{-\frac{x}{2}}$.

   b) Use your graph to estimate when $4^{-\frac{x}{2}} = 6$.

8 Money and finance

Currency conversions

In 2012, 1 euro could be exchanged for 1.25 Australian dollars (A$).

Worked examples

a) How many Australian dollars can be bought for €400?
€1 buys A$1.25.
€400 buys $1.25 \times 400 =$ A$500.

b) How much does it cost to buy A$940?
A$1.25 costs €1.
A$940 costs $\frac{1 \times 940}{1.25} =$ €752.

Exercise 8.1

The table shows the exchange rate for €1 into various currencies.

Australia	1.25 Australian dollars (A$)
India	70 rupees
Zimbabwe	470 Zimbabwe dollars (ZIM$)
South Africa	11 rand
Turkey	2.3 Turkish lira (L)
Japan	103 yen
Kuwait	0.4 dinar
USA	1.3 US dollars (US$)

1. Convert the following:
a) €25 into Australian dollars
b) €50 into rupees
c) €20 into Zimbabwe dollars
d) €300 into rand
e) €130 into Turkish lira
f) €40 into yen
g) €400 into dinar
h) €150 into US dollars

2. How many euro does it cost to buy the following:
a) A$500
b) 200 rupees
c) ZIM$1000
d) 500 rand
e) 750 Turkish lira
f) 1200 yen
g) 50 dinar
h) US$150

Earnings

Net pay is what is left after deductions such as tax, insurance and pension contributions are taken from **gross earnings**. That is, Net pay = Gross pay − Deductions

A **bonus** is an extra payment sometimes added to an employee's basic pay.

In many companies there is a fixed number of hours that an employee is expected to work. Any work done in excess of this **basic week** is paid at a higher rate, referred to as **overtime**. Overtime may be 1.5 times basic pay, called **time and a half**, or twice basic pay, called **double time**.

Piece work is another method of payment. Employees are paid for the number of articles made, not for the time taken.

Exercise 8.2

1. Mr Ahmet's gross pay is \$188.25. Deductions amount to \$33.43. What is his net pay?
2. Miss Said's basic pay is \$128. She earns \$36 for overtime and receives a bonus of \$18. What is her gross pay?
3. Mrs Hafar's gross pay is \$203. She pays \$54 in tax and \$18 towards her pension. What is her net pay?
4. Mr Wong works 35 hours for an hourly rate of \$8.30. What is his basic pay?
5. a) Miss Martinez works 38 hours for an hourly rate of \$4.15. In addition she works 6 hours of overtime at time and a half. What is her total gross pay?
   b) Deductions amount to 32% of her total gross pay. What is her net pay?
6. Pepe is paid \$5.50 for each basket of grapes he picks. One week he picks 25 baskets. How much is he paid?
7. Maria is paid €5 for every 12 plates that she makes. This is her record for one week.

Mon	240
Tues	360
Wed	288
Thurs	192
Fri	180

How much is she paid?

8. Neo works at home making clothes. The patterns and materials are provided by the company. The table shows the rates she is paid and the number of items she makes in one week:

Item	Rate	Number made
Jacket	25 rand	3
Trousers	11 rand	12
Shirt	13 rand	7
Dress	12 rand	0

a) What are her gross earnings?
b) Deductions amount to 15% of gross earnings. What is her net pay?

Profit and loss

Foodstuffs and manufactured goods are produced at a cost, known as the **cost price**, and sold at the **selling price**. If the selling price is greater than the cost price, a profit is made.

Worked example

A market trader buys oranges in boxes of 12 dozen for \$14.40 per box. He buys three boxes and sells all the oranges for 12c each. What is his profit or loss?

Cost price: $3 \times \$14.40 = \$43.20$
Selling price: $3 \times 144 \times 12c = \$51.84$
In this case he makes a profit of $\$51.84 - \$43.20$
His profit is \$8.64.

A second way of solving this problem would be:
\$14.40 for a box of 144 oranges is 10c each.
So cost price of each orange is 10c, and selling price of each orange is 12c. The profit is 2c per orange.
So 3 boxes would give a profit of $3 \times 144 \times 2c$.
That is, \$8.64.

Sometimes, particularly during sales or promotions, the selling price is reduced, this is known as a **discount**.

Worked example

In a sale a skirt usually costing \$35 is sold at a 15% discount. What is the discount?
15% of $\$35 = 0.15 \times \$35 = \$5.25$
The discount is \$5.25.

Exercise 8.3

1. A market trader buys peaches in boxes of 120. He buys 4 boxes at a cost price of \$13.20 per box. He sells 425 peaches at 12c each – the rest are ruined. How much profit or loss does he make?

2. A shopkeeper buys 72 bars of chocolate for \$5.76. What is his profit if he sells them for 12c each?

3. A holiday company charters an aircraft to fly to Malta at a cost of \$22 000. It then sells 150 seats at \$185 each and a futher 35 seats at a 20% discount. Calculate the profit made per seat if the plane has 200 seats.

4. A car is priced at \$7200. The car dealer allows a customer to pay a one-third deposit and 12 payments of \$420 per month. How much extra does it cost the customer?

5. At an auction a company sells 150 television sets for an average of \$65 each. The production cost was \$10 000. How much loss did the company make?

● Percentage profit and loss

Most profits or losses are expressed as a percentage.
Profit or loss, divided by cost price, multiplied by 100 = % profit or loss.

Worked example

A woman buys a car for \$7500 and sells it two years later for \$4500. Calculate her loss over two years as a percentage of the cost price.

cost price = \$7500   selling price = \$4500   loss = \$3000

$$\text{Loss }\% = \frac{3000}{7500} \times 100 = 40$$

Her loss is 40%.

When something becomes worth less over a period of time, it is said to **depreciate**.

Exercise 8.4

1. Find the depreciation of the following cars as a percentage of the cost price. (C.P. = cost price, S.P. = selling price)

a) VW	C.P. \$4500	S.P. \$4005
b) Rover	C.P. \$9200	S.P. \$6900

2. A company manufactures electrical items for the kitchen. Find the percentage profit on each of the following:

a) Fridge	C.P. \$50	S.P. \$65
b) Freezer	C.P. \$80	S.P. \$96

3. A developer builds a number of different types of house. Which type gives the developer the largest percentage profit?

Type A	C.P. \$40 000	S.P. \$52 000
Type B	C.P. \$65 000	S.P. \$75 000
Type C	C.P. \$81 000	S.P. \$108 000

4. Students in a school organise a disco. The disco company charges \$350 hire charge. The students sell 280 tickets at \$2.25. What is the percentage profit?

Interest

Interest can be defined as money added by a bank to sums deposited by customers. The money deposited is called the **principal**. The **percentage interest** is the given rate and the money is left for a fixed period of time.

A formula can be obtained for **simple interest**:

$$SI = \frac{Ptr}{100}$$

where SI = simple interest, i.e. the interest paid
P = the principal
t = time in years
r = rate percent

Worked examples

a) Find the simple interest earned on \$250 deposited for 6 years at 8% p.a.

$$SI = \frac{Ptr}{100}$$

$$SI = \frac{250 \times 6 \times 8}{100}$$

$$SI = 120$$

So the interest paid is \$120.

b) How long will it take for a sum of \$250 invested at 8% to earn interest of \$80?

$$SI = \frac{Ptr}{100}$$

$$80 = \frac{250 \times t \times 8}{100}$$

$$80 = 20t$$

$$4 = t$$

It will take 4 years.

c) What rate per year must be paid for a principal of \$750 to earn interest of \$180 in 4 years?

$$SI = \frac{Ptr}{100}$$

$$180 = \frac{750 \times 4 \times r}{100}$$

$$180 = 30r$$

$$6 = r$$

The rate must be 6% per year.

d) Find the principal which will earn interest of \$120 in 6 years at 4%.

$$SI = \frac{Ptr}{100}$$

$$120 = \frac{P \times 6 \times 4}{100}$$

$$120 = \frac{24P}{100}$$

$$12\,000 = 24P$$

$$500 = P$$

So the principal is \$500.

Exercise 8.5

All rates of interest given here are annual rates.

1. Find the simple interest paid in the following cases:

a) Principal \$300 rate 6% time 4 years

b) Principal \$750 rate 8% time 7 years

2. Calculate how long it will take for the following amounts of interest to be earned at the given rate.

a) $P = \$500$ $r = 6\%$ $SI = \$150$

b) $P = \$400$ $r = 9\%$ $SI = \$252$

3. Calculate the rate of interest per year which will earn the given amount of interest:

a) Principal \$400 time 4 years interest \$112

b) Principal \$800 time 7 years interest \$224

4. Calculate the principal which will earn the interest below in the given number of years at the given rate:

a) $SI = \$36$ time = 3 years rate = 6%

b) $SI = \$340$ time = 5 years rate = 8%

5. What rate of interest is paid on a deposit of \$2000 which earns \$400 interest in 5 years?

6. How long will it take a principal of \$350 to earn \$56 interest at 8% per year?

7. A principal of \$480 earns \$108 interest in 5 years. What rate of interest was being paid?

8. A principal of \$750 becomes a total of \$1320 in 8 years. What rate of interest was being paid?

9. \$1500 is invested for 6 years at 3.5% per year. What is the interest earned?

10. \$500 is invested for 11 years and becomes \$830 in total. What rate of interest was being paid?

Compound interest

Compound interest means interest is paid not only on the principal amount, but also on the interest itself: it is compounded (or added to).

This sounds complicated but the example below will make it clear.

e.g. A builder is going to build six houses on a plot of land in Spain. He borrows €500 000 at 10% interest and will pay off the loan in full after three years.

At the end of the first year he will owe:
€500 000 + 10% of €500 000 i.e. €500 000 × 1.10 = €550 000

At the end of the second year he will owe:
€550 000 + 10% of €550 000 i.e. €550 000 × 1.10 = €605 000

At the end of the third year he will owe:
€605 000 + 10% of €605 000 i.e. €605 000 × 1.10 = €665 500

The compound interest he has to pay is €665 500 − €500 000 i.e. €165 500

The time taken for a debt to grow at compound interest can be calculated as shown in the example below:

Worked example How long will it take for a debt to double at a compound interest rate of 27% p.a.?

An interest rate of 27% implies a multiplier of 1.27.

Time (years)	0	1	2	3
Debt	P	$1.27P$	$1.27^2P = 1.61P$	$1.27^3P = 2.05P$

× 1.27 × 1.27 × 1.27

The debt will have more than doubled after 3 years.

Using the example above of the builder's loan, if P represents the principal he borrows, then after 1 year his debt (D) will be given by the formula:

$D = P\left(1 + \frac{r}{100}\right)$ where r is the rate of interest.

After 2 years: $D = P\left(1 + \frac{r}{100}\right)\left(1 + \frac{r}{100}\right)$

After 3 years: $D = P\left(1 + \frac{r}{100}\right)\left(1 + \frac{r}{100}\right)\left(1 + \frac{r}{100}\right)$

After n years: $D = P\left(1 + \frac{r}{100}\right)^n$

This formula for the debt includes the original loan.
By subtracting P, the compound interest is calculated:

$$I = P\left(1 + \frac{r}{100}\right)^n - P$$

Compound interest is an example of a geometric sequence and therefore of exponential growth.

The interest is usually calculated annually, but there can be other time periods. Compound interest can be charged yearly, half-yearly, quarterly, monthly or daily. (In theory any time period can be chosen.)

Worked examples

a) Find the compound interest paid on a loan of \$600 for 3 years at an annual percentage rate (APR) of 5%.

When the rate is 5%, $1 + \frac{5}{100} = 1.05$.

$D = 600 \times 1.05^3 = 694.58$ (to 2 d.p.)

The total payment is \$694.58 so the interest due is \$694.58 − \$600 = \$94.58.

b) Find the compound interest when \$3000 is invested for 18 months at an APR of 8.5%. The interest is calculated every six months.

Note: The interest for each time period of 6 months is $\frac{8.5}{2}\% = 4.25\%$. There will therefore be 3 time periods of 6 months each.

When the rate is 4.25%, $1 + \frac{4.25}{100} = 1.0425$.

$D = 3000 \times 1.0425^3 = 3398.986 \ldots$

The final sum is \$3399, so the interest is \$3399 − \$3000 = \$399.

Exercise 8.6

1. A shipping company borrows \$70 million at 5% p.a. compound interest to build a new cruise ship. If it repays the debt after 3 years, how much interest will the company pay?
2. A woman borrows \$100 000 for home improvements. The compound interest rate is 15% p.a. and she repays it in full after 3 years. How much interest will she pay?
3. A man owes \$5000 on his credit cards. The APR is 20%. If he doesn't repay any of the debt, how much will he owe after 4 years?
4. A school increases its intake by 10% each year. If it starts with 1000 students, how many will it have at the beginning of the fourth year of expansion?

5. 8 million tonnes of fish were caught in the North Sea in 2005. If the catch is reduced by 20% each year for 4 years, what weight is caught at the end of this time?

6. How many years will it take for a debt to double at 42% p.a. compound interest?

7. How many years will it take for a debt to double at 15% p.a. compound interest?

8. A car loses value at a rate of 27% each year. How long will it take for its value to halve?

Student assessment 1

1. A visitor from Hong Kong receives 12 Pakistan rupees for each Hong Kong dollar.
   a) How many Pakistan rupees would he get for HK$240?
   b) How many Hong Kong dollars does it cost for 1 thousand rupees?

2. Below is a currency conversion table showing the amount of foreign currency received for 1 euro.

New Zealand	1.6 dollars (NZ$)
Brazil	2.6 reals

   a) How many euro does it cost for NZ$1000?
   b) How many euro does it cost for 500 Brazilian reals?

3. A girl works in a shop on Saturdays for 8.5 hours. She is paid $3.60 per hour. What is her gross pay for 4 weeks' work?

4. A potter makes cups and saucers in a factory. He is paid $1.44 per batch of cups and $1.20 per batch of saucers. What is his gross pay if he makes 9 batches of cups and 11 batches of saucers in one day?

5. Calculate the missing numbers from the simple interest table below:

Principal ($)	Rate (%)	Time (years)	Interest ($)
300	6	4	(a)
250	(b)	3	60
480	5	(c)	96
650	(d)	8	390
(e)	3.75	4	187.50

6. A family house was bought for \$48 000 twelve years ago. It is now valued at \$120 000. What is the average annual increase in the value of the house?

7. An electrician bought five broken washing machines for \$550. He repaired them and sold them for \$143 each. What was his percentage profit?

Student assessment 2

1. Find the simple interest paid on the following principal sums P, deposited in a savings account for t years at a fixed rate of interest of r%:
   a) $P = \$550$ $t = 5$ years $r = 3\%$
   b) $P = \$8000$ $t = 10$ years $r = 6\%$
   c) $P = \$12500$ $t = 7$ years $r = 2.5\%$

2. A sum of \$25 000 is deposited in a bank. After 8 years, the simple interest gained was \$7000. Calculate the annual rate of interest on the account assuming it remained constant over the 8 years.

3. A bank lends a business \$250 000. The annual rate of interest is 8.4%. When paying back the loan, the business pays an amount of \$105 000 in simple interest. Calculate the number of years the business took out the loan for.

4. Find the compound interest paid on the following principal sums P, deposited in a savings account for n years at a fixed rate of interest of r%:
   a) $P = \$400$ $n = 2$ years $r = 3\%$
   b) $P = \$5000$ $n = 8$ years $r = 6\%$
   c) $P = \$18000$ $n = 10$ years $r = 4.5\%$

5. A car is bought for \$12 500. Its value depreciates by 15% per year.
   a) Calculate its value after:
      i) 1 year ii) 2 years
   b) After how many years will the car be worth less than \$1000?

9 Time

Times may be given in terms of the 12-hour clock. We tend to say, 'I get up at seven o'clock in the morning, play football at half past two in the afternoon, and go to bed before eleven o'clock'.

These times can be written as 7 a.m., 2.30 p.m. and 11 p.m.

In order to save confusion, most timetables are written using the 24-hour clock.

7 a.m. is written as 07 00
2.30 p.m. is written as 14 30
11.00 p.m. is written as 23 00

Worked example A train covers the 480 km journey from Paris to Lyon at an average speed of 100 km/h. If the train leaves Paris at 0835, when does it arrive in Lyon?

$$\text{Time taken} = \frac{\text{distance}}{\text{speed}}$$

Paris to Lyon $= \frac{480}{100}$ hours, that is, 4.8 hours.

4.8 hours is 4 hours and (0.8 × 60 minutes), that is, 4 hours and 48 minutes.

Departure 08 35; arrival 08 35 + 04 48

Arrival time is 13 23.

Exercise 9.1

1. A journey to work takes a woman three quarters of an hour. If she catches the bus at 07 55, when does she arrive?
2. The same woman catches a bus home each evening. The journey takes 55 minutes. If she catches the bus at 17 50, when does she arrive?
3. A boy cycles to school each day. His journey takes 70 minutes. When will he arrive if he leaves home at 07 15?
4. Find the time in hours and minutes for the following journeys of the given distance at the average speed stated:
   a) 230 km at 100 km/h   b) 70 km at 50 km/h
5. Grand Prix racing cars cover a 120 km race at the following average speeds. How long do the first five cars take to complete the race? Answer in minutes and seconds.

   First 240 km/h   Second 220 km/h   Third 210 km/h
   Fourth 205 km/h   Fifth 200 km/h

6. A train covers the 1500 km distance from Amsterdam to Barcelona at an average speed of 90 km/h. If the train leaves Amsterdam at 9.30 a.m. on Tuesday, when does it arrive in Barcelona?

7. A plane takes off at 16 25 for the 3200 km journey from Moscow to Athens. If the plane flies at an average speed of 600 km/h, when will it land in Athens?

8. A plane leaves London for Boston, a distance of 5200 km, at 09 45. The plane travels at an average speed of 800 km/h. If Boston time is five hours behind British time, what is the time in Boston when the aircraft lands?

Student assessment 1

1. A journey to school takes a girl 25 minutes. What time does she arrive if she leaves home at 08 38?

2. A car travels 295 km at 50 km/h. How long does the journey take? Give your answer in hours and minutes.

3. A bus leaves Deltaville at 11 32. It travels at an average speed of 42 km/h. If it arrives in Eastwich at 12 42, what is the distance between the two towns?

4. A plane leaves Betatown at 17 58 and arrives at Fleckley at 05 03 the following morning. How long does the journey take? Give your answer in hours and minutes.

Student assessment 2

1. A journey to school takes a boy 22 minutes. What is the latest time he can leave home if he must be at school at 08 40?

2. A plane travels 270 km at 120 km/h. How long does the journey take? Give your answer in hours and minutes.

3. A train leaves Alphaville at 13 27. It travels at an average speed of 56 km/h. If it arrives in Eastwich at 16 12, what is the distance between the two towns?

4. A car leaves Gramton at 16 39. It travels a distance of 315 km and arrives at Halfield at 20 09.

   a) How long does the journey take?
   b) What is the car's average speed?

10 Set notation and Venn diagrams

Sets

A **set** is a well defined group of objects or symbols. The objects or symbols are called the **elements** of the set. If an element e belongs to a set S, this is represented as $e \in S$. If e does not belong to set S this is represented as $e \notin S$.

Worked examples

a) A particular set consists of the following elements:

{South Africa, Namibia, Egypt, Angola, ...}

i) Describe the set.

The elements of the set are countries of Africa.

ii) Add another two elements to the set.

e.g. Zimbabwe, Ghana

iii) Is the set finite or infinite?

Finite. There is a finite number of countries in Africa.

b) Consider the set $A = \{x: x \text{ is a natural number}\}$

i) Describe the set.

The elements of the set are the natural numbers.

ii) Write down two elements of the set.

e.g. 3 and 15

c) Consider the set $B = \{(x, y): y = 2x - 4\}$

i) Describe the set.

The elements of the set are the coordinates of points found on the straight line with equation $y = 2x - 4$.

ii) Write down two elements of the set.

e.g. $(0, -4)$ and $(10, 16)$

d) Consider the set $C = \{x: 2 \leq x \leq 8\}$

i) Describe the set.

The elements of the set include any number between 2 and 8 inclusive.

ii) Write down two elements of the set.

e.g. 5 and 6.3

Exercise 10.1

1. In the following questions:
   i) describe the set in words,
   ii) write down another two elements of the set.
   a) {Asia, Africa, Europe, ...}
   b) {2, 4, 6, 8, ...}
   c) {Sunday, Monday, Tuesday, ...}
   d) {January, March, July, ...}
   e) {1, 3, 6, 10, ...}
   f) {Mehmet, Michael, Mustapha, Matthew, ...}
   g) {11, 13, 17, 19, ...}
   h) {a, e, i, ...}
   i) {Earth, Mars, Venus, ...}
   j) $A = \{x: 3 \leqslant x \leqslant 12\}$
   k) $S = \{y: -5 \leqslant y \leqslant 5\}$

2. The number of elements in a set A is written as $n(A)$. Give the value of $n(A)$ for the finite sets in questions 1a–k above.

Subsets

If all the elements of one set X are also elements of another set Y, then X is said to be a **subset** of Y.

This is written as $X \subseteq Y$.

If a set A is empty (i.e. it has no elements in it), then this is called the **empty set** and it is represented by the symbol $\varnothing$. Therefore $A = \varnothing$. The empty set is a subset of all sets.

e.g. Three girls, Winnie, Natalie and Emma form a set A
A = {Winnie, Natalie, Emma}
All the possible subsets of A are given below:
B = {Winnie, Natalie, Emma}
C = {Winnie, Natalie}
D = {Winnie, Emma}
E = {Natalie, Emma}
F = {Winnie}
G = {Natalie}
H = {Emma}
$I = \varnothing$

Note that the sets B and I above are considered as subsets of A.

i.e. $B \subseteq A$ and $I \subseteq A$

However, sets C, D, E, F, G and H are considered **proper subsets** of A. This distinction of subset is shown in the notation below.

$C \subset A$ and $D \subset A$ etc.

Similarly $G \nsubseteq H$ implies that G is not a subset of H

$G \not\subset H$ implies that G is not a proper subset of H

Worked example $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

i) List subset B {even numbers}.

$B = \{2, 4, 6, 8, 10\}$

ii) List subset C {prime numbers}.

$C = \{2, 3, 5, 7\}$

Exercise 10.2

1. P = {whole numbers less than 30}
   a) List the subset Q {even numbers}.
   b) List the subset R {odd numbers}.
   c) List the subset S {prime numbers}.
   d) List the subset T {square numbers}.
   e) List the subset U {triangle numbers}.

2. A = {whole numbers between 50 and 70}
   a) List the subset B {multiples of 5}.
   b) List the subset C {multiples of 3}.
   c) List the subset D {square numbers}.

3. $J = \{p, q, r\}$
   a) List all the subsets of J.
   b) List all the proper subsets of J.

4. State whether each of the following statements is true or false:
   a) {Algeria, Mozambique} $\subseteq$ {countries in Africa}
   b) {mango, banana} $\subseteq$ {fruit}
   c) $\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}$
   d) $\{1, 2, 3, 4\} \subset \{1, 2, 3, 4\}$
   e) {volleyball, basketball} $\not\subseteq$ {team sport}
   f) $\{4, 6, 8, 10\} \not\subset \{4, 6, 8, 10\}$
   g) {potatoes, carrots} $\subseteq$ {vegetables}
   h) $\{12, 13, 14, 15\} \not\subset$ {whole numbers}

● The universal set

The **universal set** ($\mathscr{E}$) for any particular problem is the set which contains all the possible elements for that problem.

The **complement** of a set A is the set of elements which are in $\mathscr{E}$ but not in A. The complement of A is identified as A'. Notice that $\mathscr{E}' = \varnothing$ and $\varnothing' = \mathscr{E}$.

Worked examples

a) If $\mathscr{E} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 2, 3, 4, 5\}$ what set is represented by A'?

A' consists of those elements in $\mathscr{E}$ which are not in A.

Therefore $A' = \{6, 7, 8, 9, 10\}$.

b) If $\mathscr{E}$ = {all 3D shapes} and P = {prisms} what set is represented by P'?

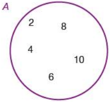
P' = {all 3D shapes except prisms}.

Set notation and Venn diagrams

Venn diagrams are the principal way of showing sets diagrammatically. The method consists primarily of entering the elements of a set into a circle or circles.

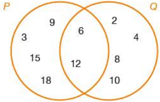
Some examples of the uses of Venn diagrams are shown.

$A = \{2, 4, 6, 8, 10\}$ can be represented as:



Elements which are in more than one set can also be represented using a Venn diagram.

$P = \{3, 6, 9, 12, 15, 18\}$ and $Q = \{2, 4, 6, 8, 10, 12\}$ can be represented as:

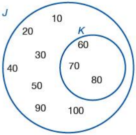


In the diagram above it can be seen that those elements which belong to both sets are placed in the region of overlap of the two circles.

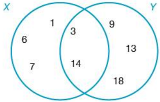
When two sets P and Q overlap as they do above, the notation $P \cap Q$ is used to denote the set of elements in the **intersection**, i.e. $P \cap Q = \{6, 12\}$.

Note that $6 \in P \cap Q$; $8 \notin P \cap Q$.

$J = \{10, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$ and $K = \{60, 70, 80\}$; as discussed earlier, $K \subset J$ can be represented as shown below:



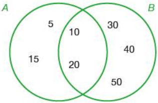
$X = \{1, 3, 6, 7, 14\}$ and $Y = \{3, 9, 13, 14, 18\}$ are represented as:



The **union** of two sets is everything which belongs to either or both sets and is represented by the symbol $\cup$.

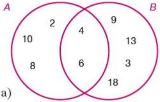
Therefore in the example above $X \cup Y = \{1, 3, 6, 7, 9, 13, 14, 18\}$.

Exercise 10.3

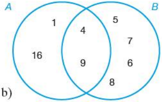


1. Using the Venn diagram (left), indicate whether the following statements are true or false. $\in$ means 'is an element of' and $\notin$ means 'is not an element of'.
   a) $5 \in A$ b) $20 \in B$ c) $20 \notin A$
   d) $50 \in A$ e) $50 \notin B$ f) $A \cap B = \{10, 20\}$

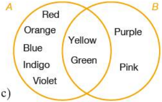
2. Complete the statement $A \cap B = \{\ldots\}$ for each of the Venn diagrams below.



a)



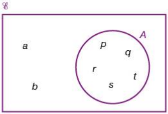
b)



c)

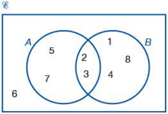
3. Complete the statement $A \cup B = \{\ldots\}$ for each of the Venn diagrams in question 2 above.

4.



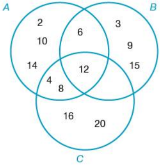
Copy and complete the following statements:
a) $\mathscr{E} = \{\ldots\}$ b) $A' = \{\ldots\}$

5.



Copy and complete the following statements:

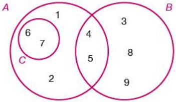
a) $\mathscr{E} = \{\ldots\}$ b) $A' = \{\ldots\}$ c) $A \cap B = \{\ldots\}$
d) $A \cup B = \{\ldots\}$ e) $(A \cap B)' = \{\ldots\}$ f) $A \cap B' = \{\ldots\}$



6. a) Describe in words the elements of:
i) set A ii) set B iii) set C

b) Copy and complete the following statements:
i) $A \cap B = \{\ldots\}$ ii) $A \cap C = \{\ldots\}$
iii) $B \cap C = \{\ldots\}$ iv) $A \cap B \cap C = \{\ldots\}$
v) $A \cup B = \{\ldots\}$ vi) $C \cup B = \{\ldots\}$

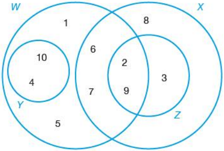
7.



a) Copy and complete the following statements:
i) $A = \{\ldots\}$ ii) $B = \{\ldots\}$
iii) $C' = \{\ldots\}$ iv) $A \cap B = \{\ldots\}$
v) $A \cup B = \{\ldots\}$ vi) $(A \cap B)' = \{\ldots\}$

b) State, using set notation, the relationship between C and A.

8.



a) Copy and complete the following statements:
i) $W = \{\ldots\}$ ii) $X = \{\ldots\}$
iii) $Z' = \{\ldots\}$ iv) $W \cap Z = \{\ldots\}$
v) $W \cap X = \{\ldots\}$ vi) $Y \cap Z = \{\ldots\}$

b) Which of the named sets is a subset of X?

Exercise 10.4

1. $A = \{\text{Egypt, Libya, Morocco, Chad}\}$
   $B = \{\text{Iran, Iraq, Turkey, Egypt}\}$
   a) Draw a Venn diagram to illustrate the above information.
   b) Copy and complete the following statements:
      i) $A \cap B = \{\ldots\}$ ii) $A \cup B = \{\ldots\}$

2. $P = \{2, 3, 5, 7, 11, 13, 17\}$
   $Q = \{11, 13, 15, 17, 19\}$
   a) Draw a Venn diagram to illustrate the above information.
   b) Copy and complete the following statements:
      i) $P \cap Q = \{\ldots\}$ ii) $P \cup Q = \{\ldots\}$

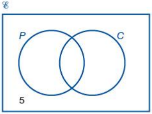
3. $B = \{2, 4, 6, 8, 10\}$
   $A \cup B = \{1, 2, 3, 4, 6, 8, 10\}$
   $A \cap B = \{2, 4\}$
   Represent the above information on a Venn diagram.

4. $X = \{a, c, d, e, f, g, l\}$
   $Y = \{b, c, d, e, h, i, k, l, m\}$
   $Z = \{c, f, i, j, m\}$
   Represent the above information on a Venn diagram.

5. $P = \{1, 4, 7, 9, 11, 15\}$
   $Q = \{5, 10, 15\}$
   $R = \{1, 4, 9\}$
   Represent the above information on a Venn diagram.

● Problems involving sets

Worked example



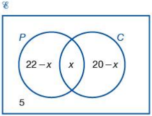
In a class of 31 students, some study Physics and some study Chemistry. If 22 study Physics, 20 study Chemistry and 5 study neither, calculate the number of students who take both subjects.

The information given above can be entered in a Venn diagram in stages.

The students taking neither Physics nor Chemistry can be put in first (as shown left).

This leaves 26 students to be entered into the set circles.

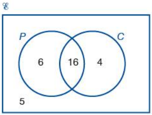
If x students take both subjects then



$$n(P) = 22 - x + x$$
$$n(C) = 20 - x + x$$
$$P \cup C = 31 - 5 = 26$$

Therefore
$$22 - x + x + 20 - x = 26$$
$$42 - x = 26$$
$$x = 16$$

Substituting the value of x into the Venn diagram gives:



Therefore the number of students taking both Physics and Chemistry is 16.

Exercise 10.5

1. In a class of 35 students, 19 take Spanish, 18 take French and 3 take neither. Calculate how many take:
   a) both French and Spanish,
   b) just Spanish,
   c) just French.

2. In a year group of 108 students, 60 liked football, 53 liked tennis and 10 liked neither. Calculate the number of students who liked football but not tennis.

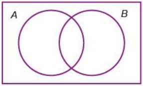
3. In a year group of 113 students, 60 liked hockey, 45 liked rugby and 18 liked neither. Calculate the number of students who:
   a) liked both hockey and rugby,
   b) liked only hockey.

4. One year 37 students sat an examination in Physics, 48 sat Chemistry and 45 sat Biology. 15 students sat Physics and Chemistry, 13 sat Chemistry and Biology, 7 sat Physics and Biology and 5 students sat all three.
   a) Draw a Venn diagram to represent this information.
   b) Calculate $n\,(P \cup C \cup B)$.

Student assessment 1

1. Describe the following sets in words:
   a) $\{2, 4, 6, 8\}$
   b) $\{2, 4, 6, 8, \ldots\}$
   c) $\{1, 4, 9, 16, 25, \ldots\}$
   d) {Arctic, Atlantic, Indian, Pacific}

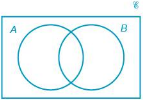
2. Calculate the value of $n(A)$ for each of the sets shown below:
   a) $A =$ {days of the week}
   b) $A =$ {prime numbers between 50 and 60}
   c) $A = \{x : x \text{ is an integer and } 5 \leqslant x \leqslant 10\}$
   d) $A =$ {days in a leap year}



3. Copy out the Venn diagram (left) twice.
   a) On one copy shade and label the region which represents $A \cap B$.
   b) On the other copy shade and label the region which represents $A \cup B$.
4. If $A = \{a, b\}$ list all the subsets of A.
5. If $\mathscr{E} = \{m, a, t, h, s\}$ and $A = \{a, s\}$, what set is represented by A'?

Student assessment 2

1. Describe the following sets in words:
   a) $\{1, 3, 5, 7\}$
   b) $\{1, 3, 5, 7, \ldots\}$
   c) $\{1, 3, 6, 10, 15, \ldots\}$
   d) {Brazil, Chile, Argentina, Bolivia, ...}
2. Calculate the value of $n(A)$ for each of the sets shown below:
   a) A = {months of the year}
   b) A = {square numbers between 99 and 149}
   c) $A = \{x: x \text{ is an integer and } -9 \leqslant x \leqslant -3\}$
   d) A = {students in your class}



3. Copy out the Venn diagram (left) twice.
   a) On one copy shade and label the region which represents $\mathscr{E}$.
   b) On the other copy shade and label the region which represents $(A \cap B)'$.
4. If $A = \{w, o, r, k\}$ list all the subsets of A with at least three elements.
5. If $\mathscr{E} = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $P = \{2, 4, 6, 8\}$, what set is represented by P'?

Student assessment 3

1. If $A = \{2, 4, 6, 8\}$ write all the proper subsets of A with two or more elements.
2. J = {London, Paris, Rome, Washington, Canberra, Ankara, Cairo}
   K = {Cairo, Nairobi, Pretoria, Ankara}
   a) Draw a Venn diagram to represent the above information.
   b) Copy and complete the statement $J \cap K = \{\ldots\}$.
   c) Copy and complete the statement $J' \cap K = \{\ldots\}$.

3. $M = \{x: x \text{ is an integer and } 2 \leqslant x \leqslant 20\}$
   $N = \{\text{prime numbers less than } 30\}$
   a) Draw a Venn diagram to illustrate the information above.
   b) Copy and complete the statement $M \cap N = \{\ldots\}$.
   c) Copy and complete the statement $(M \cap N)' = \{\ldots\}$.

4. $\mathscr{E} = \{\text{natural numbers}\}$, $M = \{\text{even numbers}\}$ and $N = \{\text{multiples of } 5\}$.
   a) Draw a Venn diagram and place the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 in the appropriate places in it.
   b) If $X = M \cap N$, describe set X in words.

5. In a region of mixed farming, farms keep goats, cattle or sheep. There are 77 farms altogether. 19 farms keep only goats, 8 keep only cattle and 13 keep only sheep. 13 keep both goats and cattle, 28 keep both cattle and sheep and 8 keep both goats and sheep.
   a) Draw a Venn diagram to show the above information.
   b) Calculate $n(G \cap C \cap S)$.

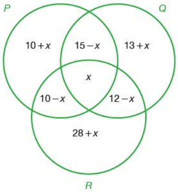
Student assessment 4

1. $M = \{a, e, i, o, u\}$
   a) How many subsets are there of M?
   b) List the subsets of M with four or more elements.

2. X = {lion, tiger, cheetah, leopard, puma, jaguar, cat}
   Y = {elephant, lion, zebra, cheetah, gazelle}
   Z = {anaconda, jaguar, tarantula, mosquito}
   a) Draw a Venn diagram to represent the above information.
   b) Copy and complete the statement $X \cap Y = \{\ldots\}$.
   c) Copy and complete the statement $Y \cap Z = \{\ldots\}$.
   d) Copy and complete the statement $X \cap Y \cap Z = \{\ldots\}$.

3. A group of 40 people were asked whether they like cricket (C) and football (F). The number liking both cricket and football was three times the number liking only cricket. Adding 3 to the number liking only cricket and doubling the answer equals the number of people liking only football. Four said they did not like sport at all.
   a) Draw a Venn diagram to represent this information.
   b) Calculate $n(C \cap F)$.
   c) Calculate $n(C \cap F')$.
   d) Calculate $n(C' \cap F)$.

4. The Venn diagram below shows the number of elements in three sets P, Q and R.



If $n(P \cup Q \cup R) = 93$ calculate:

a) x b) $n(P)$ c) $n(Q)$
d) $n(R)$ e) $n(P \cap Q)$ f) $n(Q \cap R)$
g) $n(P \cap R)$ h) $n(R \cup Q)$ i) $n(P \cap Q)'$

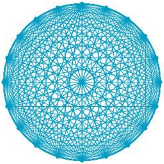
Topic 1 Mathematical investigations and ICT

Investigations are an important part of mathematical learning. All mathematical discoveries stem from an idea that a mathematician has and then investigates.

Sometimes when faced with a mathematical investigation, it can seem difficult to know how to start. The structure and example below may help you.

1. Read the question carefully and start with simple cases.
2. Draw simple diagrams to help.
3. Put the results from simple cases in a table.
4. Look for a pattern in your results.
5. Try to find a general rule in words.
6. Express your rule algebraically.
7. Test the rule for a new example.
8. Check that the original question has been answered.

Worked example



A mystic rose is created by placing a number of points evenly spaced on the circumference of a circle. Straight lines are then drawn from each point to every other point. The diagram (left) shows a mystic rose with 20 points.

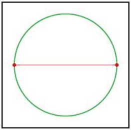
i) How many straight lines are there?
ii) How many straight lines would there be on a mystic rose with 100 points?

To answer these questions, you are not expected to draw either of the shapes and count the number of lines.

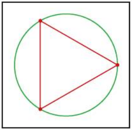
1/2. Try simple cases:

By drawing some simple cases and counting the lines, some results can be found:

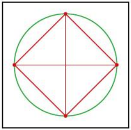
Mystic rose with 2 points
Number of lines = 1



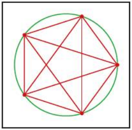
Mystic rose with 3 points
Number of lines = 3



Mystic rose with 4 points
Number of lines = 6



Mystic rose with 5 points
Number of lines = 10

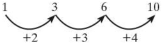


3. Enter the results in an ordered table:

Number of points	2	3	4	5
Number of lines	1	3	6	10

4/5. Look for a pattern in the results:

There are two patterns.
The first shows how the values change.



It can be seen that the difference between successive terms is increasing by one each time.

The problem with this pattern is that to find the 20th and 100th terms, it would be necessary to continue this pattern and find all the terms leading up to the 20th and 100th term.

The second is the relationship between the number of points and the number of lines.

Number of points	2	3	4	5
Number of lines	1	3	6	10

It is important to find a relationship that works for all values, for example subtracting 1 from the number of points gives the number of lines in the first example only, so is not useful. However, halving the number of points and multiplying this by 1 less than the number of points works each time,

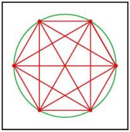
i.e. Number of lines = (half the number of points) × (one less than the number of points).

6. Express the rule algebraically:

The rule expressed in words above can be written more elegantly using algebra. Let the number of lines be l and the number of points be p.

$$l = \frac{1}{2}p(p - 1)$$

Note: Any letters can be used to represent the number of lines and the number of points, not just l and p.



7. **Test the rule:**
The rule was derived from the original results. It can be tested by generating a further result.

If the number of points $p = 6$, then the number of lines l is:

$$\begin{aligned} l &= \tfrac{1}{2} \times 6(6 - 1) \\ &= 3 \times 5 \\ &= 15 \end{aligned}$$

From the diagram to the left, the number of lines can also be counted as 15.

8. **Check that the original questions have been answered:**
Using the formula, the number of lines in a mystic rose with 20 points is:

$$\begin{aligned} l &= \tfrac{1}{2} \times 20(20 - 1) \\ &= 10 \times 19 \\ &= 190 \end{aligned}$$

The number of lines in a mystic rose with 100 points is:

$$\begin{aligned} l &= \tfrac{1}{2} \times 100(100 - 1) \\ &= 50 \times 99 \\ &= 4950 \end{aligned}$$

Primes and squares

13, 41 and 73 are prime numbers.

Two different square numbers can be added together to make these prime numbers, e.g. $3^2 + 8^2 = 73$.

1. Find the two square numbers that can be added to make 13 and 41.
2. List the prime numbers less than 100.
3. Which of the prime numbers less than 100 can be shown to be the sum of two different square numbers?
4. Is there a rule to the numbers in question 3?
5. Your rule is a predictive rule not a formula. Discuss the difference.

Football leagues

There are 18 teams in a football league.

1. If each team plays the other teams twice, once at home and once away, then how many matches are played in a season?
2. If there are t teams in a league, how many matches are played in a season?

ICT activity 1

In this activity you will be using a spreadsheet to track the price of a company's shares over a period of time.

	A	B	C
1	Company Name		
2	Day	Share Price	Percentage Value
3	1	3.26	100
4	2	3.29	
5	3	4.11	
6	4		
7	5		
8			
9			
10			
11	etc	etc	

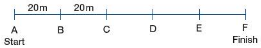
1. a) Using the internet or a newspaper as a resource, find the value of a particular company's shares.
   b) Over a period of a month (or week), record the value of the company's shares. This should be carried out on a daily basis.
2. When you have collected all the results, enter them into a spreadsheet similar to the one shown on the left.
3. In column C enter formulae that will calculate the value of the shares as a percentage of their value on day 1.
4. When the spreadsheet is complete, produce a graph showing how the percentage value of the share price changed over time.
5. Write a short report explaining the performance of the company's shares during that time.

ICT activity 2

The following activity refers to the graphing package Autograph; however, a similar package may be used.

The velocity of a student at different parts of a 100 m sprint will be analysed.

A racecourse is set out as shown below:



1. A student must stand at each of points A–F. The student at A runs the 100 m and is timed as he/she runs past each of the points B–F by the students at these points who each have a stopwatch.
2. In Autograph, plot a distance–time graph of the results by entering the data as pairs of coordinates, i.e. (time, distance).
3. Ensure that all the points are selected and draw a curve of best fit through them.
4. Select the curve and plot a coordinate of your choice on it. This point can now be moved along the curve using the cursor keys on the keyboard.
5. Draw a tangent to the curve through the point.
6. What does the gradient of the tangent represent?
7. At what point of the race was the student running fastest? How did you reach this answer?
8. Collect similar data for other students. Compare their graphs and running speeds.
9. Carefully analyse one of the graphs and write a brief report to the runner in which you should identify, giving reasons, the parts of the race he/she needs to improve on.