

STEWART
JAMES

Calculus **7E** *Early Transcendentals*

Cut here and keep for reference

ALGEBRA

Arithmetic Operations

$$a(b + c) = ab + ac$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a + c}{b} = \frac{a}{b} + \frac{c}{b}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Exponents and Radicals

$$x^m x^n = x^{m+n}$$

$$\frac{x^m}{x^n} = x^{m-n}$$

$$(x^m)^n = x^{mn}$$

$$x^{-n} = \frac{1}{x^n}$$

$$(xy)^n = x^n y^n$$

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$x^{1/n} = \sqrt[n]{x}$$

$$x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$$

$$\sqrt[n]{xy} = \sqrt[n]{x} \sqrt[n]{y}$$

$$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

Factoring Special Polynomials

$$x^2 - y^2 = (x + y)(x - y)$$

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Binomial Theorem

$$(x + y)^2 = x^2 + 2xy + y^2 \quad (x - y)^2 = x^2 - 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2$$

$$+ \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + nxy^{n-1} + y^n$$

$$\text{where } \binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{1 \cdot 2 \cdot 3 \cdots k}$$

Quadratic Formula

$$\text{If } ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Inequalities and Absolute Value

If $a < b$ and $b < c$, then $a < c$.

If $a < b$, then $a + c < b + c$.

If $a < b$ and $c > 0$, then $ca < cb$.

If $a < b$ and $c < 0$, then $ca > cb$.

If $a > 0$, then

$$|x| = a \text{ means } x = a \text{ or } x = -a$$

$$|x| < a \text{ means } -a < x < a$$

$$|x| > a \text{ means } x > a \text{ or } x < -a$$

GEOMETRY

Geometric Formulas

Formulas for area A , circumference C , and volume V :

Triangle

$$A = \frac{1}{2}bh$$

$$= \frac{1}{2}ab \sin \theta$$

Circle

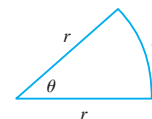
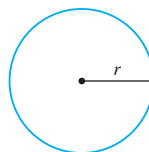
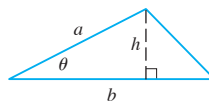
$$A = \pi r^2$$

$$C = 2\pi r$$

Sector of Circle

$$A = \frac{1}{2}r^2\theta$$

$$s = r\theta \quad (\theta \text{ in radians})$$



Sphere

$$V = \frac{4}{3}\pi r^3$$

$$A = 4\pi r^2$$

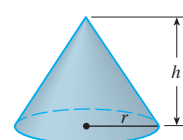
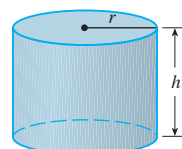
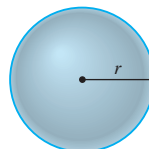
Cylinder

$$V = \pi r^2 h$$

Cone

$$V = \frac{1}{3}\pi r^2 h$$

$$A = \pi r \sqrt{r^2 + h^2}$$



Distance and Midpoint Formulas

Distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{Midpoint of } \overline{P_1P_2}: \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Lines

Slope of line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-slope equation of line through $P_1(x_1, y_1)$ with slope m :

$$y - y_1 = m(x - x_1)$$

Slope-intercept equation of line with slope m and y-intercept b :

$$y = mx + b$$

Circles

Equation of the circle with center (h, k) and radius r :

$$(x - h)^2 + (y - k)^2 = r^2$$

TRIGONOMETRY

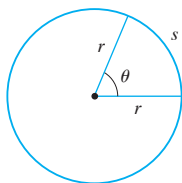
Angle Measurement

$$\pi \text{ radians} = 180^\circ$$

$$1^\circ = \frac{\pi}{180} \text{ rad} \quad 1 \text{ rad} = \frac{180^\circ}{\pi}$$

$$s = r\theta$$

(θ in radians)

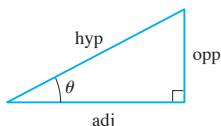


Right Angle Trigonometry

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} \quad \csc \theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} \quad \sec \theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan \theta = \frac{\text{opp}}{\text{adj}} \quad \cot \theta = \frac{\text{adj}}{\text{opp}}$$

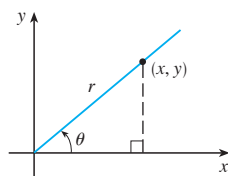


Trigonometric Functions

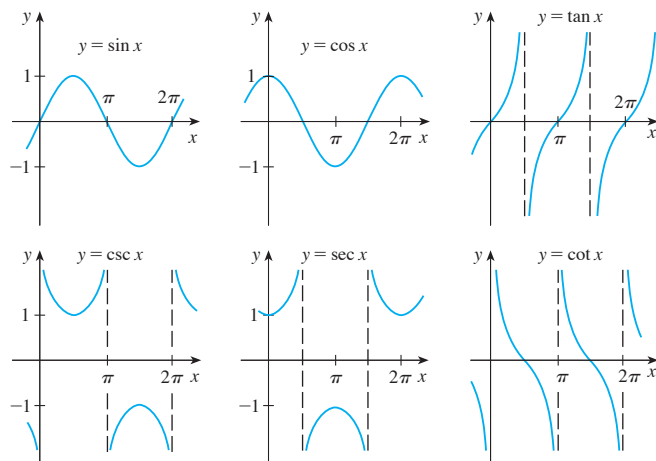
$$\sin \theta = \frac{y}{r} \quad \csc \theta = \frac{r}{y}$$

$$\cos \theta = \frac{x}{r} \quad \sec \theta = \frac{r}{x}$$

$$\tan \theta = \frac{y}{x} \quad \cot \theta = \frac{x}{y}$$



Graphs of Trigonometric Functions



Trigonometric Functions of Important Angles

θ	radians	$\sin \theta$	$\cos \theta$	$\tan \theta$
0°	0	0	1	0
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90°	$\pi/2$	1	0	—

Fundamental Identities

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\sin(-\theta) = -\sin \theta$$

$$\tan(-\theta) = -\tan \theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

$$\cos(-\theta) = \cos \theta$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$$

The Law of Sines

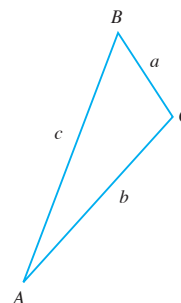
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$



Addition and Subtraction Formulas

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Double-Angle Formulas

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Half-Angle Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

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CALCULUS

EARLY TRANSCENDENTALS

SEVENTH EDITION

JAMES STEWART

McMASTER UNIVERSITY

AND

UNIVERSITY OF TORONTO



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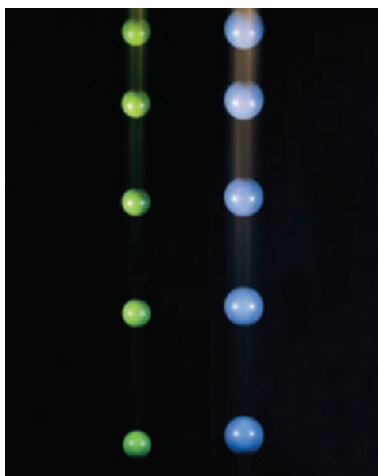
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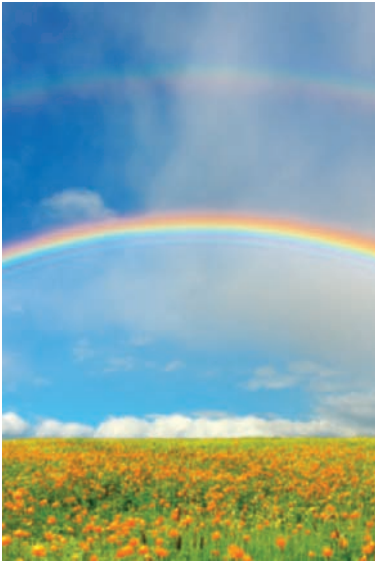
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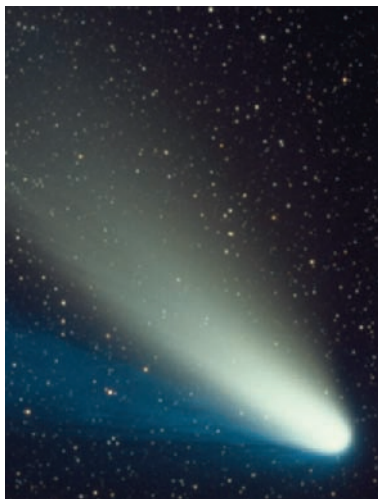
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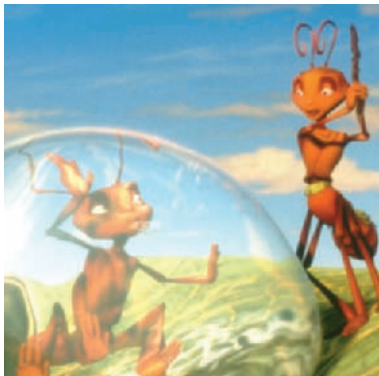
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Preface

A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery.

GEORGE POLYA

The art of teaching, Mark Van Doren said, is the art of assisting discovery. I have tried to write a book that assists students in discovering calculus—both for its practical power and its surprising beauty. In this edition, as in the first six editions, I aim to convey to the student a sense of the utility of calculus and develop technical competence, but I also strive to give some appreciation for the intrinsic beauty of the subject. Newton undoubtedly experienced a sense of triumph when he made his great discoveries. I want students to share some of that excitement.

The emphasis is on understanding concepts. I think that nearly everybody agrees that this should be the primary goal of calculus instruction. In fact, the impetus for the current calculus reform movement came from the Tulane Conference in 1986, which formulated as their first recommendation:

Focus on conceptual understanding.

I have tried to implement this goal through the *Rule of Three*: “Topics should be presented geometrically, numerically, and algebraically.” Visualization, numerical and graphical experimentation, and other approaches have changed how we teach conceptual reasoning in fundamental ways. The Rule of Three has been expanded to become the *Rule of Four* by emphasizing the verbal, or descriptive, point of view as well.

In writing the seventh edition my premise has been that it is possible to achieve conceptual understanding and still retain the best traditions of traditional calculus. The book contains elements of reform, but within the context of a traditional curriculum.



Alternative Versions

I have written several other calculus textbooks that might be preferable for some instructors. Most of them also come in single variable and multivariable versions.

- *Calculus: Early Transcendentals*, Seventh Edition, Hybrid Version, is similar to the present textbook in content and coverage except that all end-of-section exercises are available only in Enhanced WebAssign. The printed text includes all end-of-chapter review material.
- *Calculus*, Seventh Edition, is similar to the present textbook except that the exponential, logarithmic, and inverse trigonometric functions are covered in the second semester.

- *Calculus*, Seventh Edition, Hybrid Version, is similar to *Calculus*, Seventh Edition, in content and coverage except that all end-of-section exercises are available only in Enhanced WebAssign. The printed text includes all end-of-chapter review material.
- *Essential Calculus* is a much briefer book (800 pages), though it contains almost all of the topics in *Calculus*, Seventh Edition. The relative brevity is achieved through briefer exposition of some topics and putting some features on the website.
- *Essential Calculus: Early Transcendentals* resembles *Essential Calculus*, but the exponential, logarithmic, and inverse trigonometric functions are covered in Chapter 3.
- *Calculus: Concepts and Contexts*, Fourth Edition, emphasizes conceptual understanding even more strongly than this book. The coverage of topics is not encyclopedic and the material on transcendental functions and on parametric equations is woven throughout the book instead of being treated in separate chapters.
- *Calculus: Early Vectors* introduces vectors and vector functions in the first semester and integrates them throughout the book. It is suitable for students taking Engineering and Physics courses concurrently with calculus.
- *Brief Applied Calculus* is intended for students in business, the social sciences, and the life sciences.

What's New in the Seventh Edition?

The changes have resulted from talking with my colleagues and students at the University of Toronto and from reading journals, as well as suggestions from users and reviewers. Here are some of the many improvements that I've incorporated into this edition:

- Some material has been rewritten for greater clarity or for better motivation. See, for instance, the introduction to maximum and minimum values on page 274, the introduction to series on page 703, and the motivation for the cross product on page 808.
- New examples have been added (see Example 4 on page 1021 for instance). And the solutions to some of the existing examples have been amplified. A case in point: I added details to the solution of Example 2.3.11 because when I taught Section 2.3 from the sixth edition I realized that students need more guidance when setting up inequalities for the Squeeze Theorem.
- The art program has been revamped: New figures have been incorporated and a substantial percentage of the existing figures have been redrawn.
- The data in examples and exercises have been updated to be more timely.
- Three new projects have been added: *The Gini Index* (page 429) explores how to measure income distribution among inhabitants of a given country and is a nice application of areas between curves. (I thank Klaus Volpert for suggesting this project.) *Families of Implicit Curves* (page 217) investigates the changing shapes of implicitly defined curves as parameters in a family are varied. *Families of Polar Curves* (page 664) exhibits the fascinating shapes of polar curves and how they evolve within a family.
- The section on the surface area of the graph of a function of two variables has been restored as Section 15.6 for the convenience of instructors who like to teach it after double integrals, though the full treatment of surface area remains in Chapter 16.

- I continue to seek out examples of how calculus applies to so many aspects of the real world. On page 909 you will see beautiful images of the earth's magnetic field strength and its second vertical derivative as calculated from Laplace's equation. I thank Roger Watson for bringing to my attention how this is used in geophysics and mineral exploration.
- More than 25% of the exercises in each chapter are new. Here are some of my favorites: 1.6.58, 2.6.51, 2.8.13–14, 3.3.56, 3.4.67, 3.5.69–72, 3.7.22, 4.3.86, 5.2.51–53, 6.4.30, 11.2.49–50, 11.10.71–72, 12.1.44, 12.4.43–44, and Problems 4, 5, and 8 on pages 837–38.

Technology Enhancements

- The media and technology to support the text have been enhanced to give professors greater control over their course, to provide extra help to deal with the varying levels of student preparedness for the calculus course, and to improve support for conceptual understanding. New Enhanced WebAssign features including a customizable Cengage YouBook, *Just in Time* review, *Show Your Work*, Answer Evaluator, Personalized Study Plan, Master Its, solution videos, lecture video clips (with associated questions), and *Visualizing Calculus* (TEC animations with associated questions) have been developed to facilitate improved student learning and flexible classroom teaching.
- *Tools for Enriching Calculus* (TEC) has been completely redesigned and is accessible in Enhanced WebAssign, CourseMate, and PowerLecture. Selected Visuals and Modules are available at www.stewartcalculus.com.

Features

CONCEPTUAL EXERCISES

The most important way to foster conceptual understanding is through the problems that we assign. To that end I have devised various types of problems. Some exercise sets begin with requests to explain the meanings of the basic concepts of the section. (See, for instance, the first few exercises in Sections 2.2, 2.5, 11.2, 14.2, and 14.3.) Similarly, all the review sections begin with a Concept Check and a True-False Quiz. Other exercises test conceptual understanding through graphs or tables (see Exercises 2.7.17, 2.8.35–40, 2.8.43–46, 9.1.11–13, 10.1.24–27, 11.10.2, 13.2.1–2, 13.3.33–39, 14.1.1–2, 14.1.32–42, 14.3.3–10, 14.6.1–2, 14.7.3–4, 15.1.5–10, 16.1.11–18, 16.2.17–18, and 16.3.1–2).

Another type of exercise uses verbal description to test conceptual understanding (see Exercises 2.5.10, 2.8.58, 4.3.63–64, and 7.8.67). I particularly value problems that combine and compare graphical, numerical, and algebraic approaches (see Exercises 2.6.39–40, 3.7.27, and 9.4.2).

GRADED EXERCISE SETS

Each exercise set is carefully graded, progressing from basic conceptual exercises and skill-development problems to more challenging problems involving applications and proofs.



REAL-WORLD DATA

My assistants and I spent a great deal of time looking in libraries, contacting companies and government agencies, and searching the Internet for interesting real-world data to introduce, motivate, and illustrate the concepts of calculus. As a result, many of the examples and exercises deal with functions defined by such numerical data or graphs. See, for instance, Figure 1 in Section 1.1 (seismograms from the Northridge earthquake), Exercise 2.8.36 (percentage of the population under age 18), Exercise 5.1.16 (velocity of the space

shuttle *Endeavour*), and Figure 4 in Section 5.4 (San Francisco power consumption). Functions of two variables are illustrated by a table of values of the wind-chill index as a function of air temperature and wind speed (Example 2 in Section 14.1). Partial derivatives are introduced in Section 14.3 by examining a column in a table of values of the heat index (perceived air temperature) as a function of the actual temperature and the relative humidity. This example is pursued further in connection with linear approximations (Example 3 in Section 14.4). Directional derivatives are introduced in Section 14.6 by using a temperature contour map to estimate the rate of change of temperature at Reno in the direction of Las Vegas. Double integrals are used to estimate the average snowfall in Colorado on December 20–21, 2006 (Example 4 in Section 15.1). Vector fields are introduced in Section 16.1 by depictions of actual velocity vector fields showing San Francisco Bay wind patterns.

PROJECTS One way of involving students and making them active learners is to have them work (perhaps in groups) on extended projects that give a feeling of substantial accomplishment when completed. I have included four kinds of projects: *Applied Projects* involve applications that are designed to appeal to the imagination of students. The project after Section 9.3 asks whether a ball thrown upward takes longer to reach its maximum height or to fall back to its original height. (The answer might surprise you.) The project after Section 14.8 uses Lagrange multipliers to determine the masses of the three stages of a rocket so as to minimize the total mass while enabling the rocket to reach a desired velocity. *Laboratory Projects* involve technology; the one following Section 10.2 shows how to use Bézier curves to design shapes that represent letters for a laser printer. *Writing Projects* ask students to compare present-day methods with those of the founders of calculus—Fermat’s method for finding tangents, for instance. Suggested references are supplied. *Discovery Projects* anticipate results to be discussed later or encourage discovery through pattern recognition (see the one following Section 7.6). Others explore aspects of geometry: tetrahedra (after Section 12.4), hyperspheres (after Section 15.7), and intersections of three cylinders (after Section 15.8). Additional projects can be found in the *Instructor’s Guide* (see, for instance, Group Exercise 5.1: Position from Samples).

PROBLEM SOLVING Students usually have difficulties with problems for which there is no single well-defined procedure for obtaining the answer. I think nobody has improved very much on George Polya’s four-stage problem-solving strategy and, accordingly, I have included a version of his problem-solving principles following Chapter 1. They are applied, both explicitly and implicitly, throughout the book. After the other chapters I have placed sections called *Problems Plus*, which feature examples of how to tackle challenging calculus problems. In selecting the varied problems for these sections I kept in mind the following advice from David Hilbert: “A mathematical problem should be difficult in order to entice us, yet not inaccessible lest it mock our efforts.” When I put these challenging problems on assignments and tests I grade them in a different way. Here I reward a student significantly for ideas toward a solution and for recognizing which problem-solving principles are relevant.

TECHNOLOGY The availability of technology makes it not less important but more important to clearly understand the concepts that underlie the images on the screen. But, when properly used, graphing calculators and computers are powerful tools for discovering and understanding those concepts. This textbook can be used either with or without technology and I use two special symbols to indicate clearly when a particular type of machine is required. The icon  indicates an exercise that definitely requires the use of such technology, but that is not to say that it can’t be used on the other exercises as well. The symbol  is reserved for problems in which the full resources of a computer algebra system (like Derive, Maple, Mathematica, or the TI-89/92) are required. But technology doesn’t make pencil and paper

obsolete. Hand calculation and sketches are often preferable to technology for illustrating and reinforcing some concepts. Both instructors and students need to develop the ability to decide where the hand or the machine is appropriate.

TOOLS FOR ENRICHING™ CALCULUS

TEC is a companion to the text and is intended to enrich and complement its contents. (It is now accessible in Enhanced WebAssign, CourseMate, and PowerLecture. Selected Visuals and Modules are available at www.stewartcalculus.com.) Developed by Harvey Keynes, Dan Clegg, Hubert Hohn, and myself, TEC uses a discovery and exploratory approach. In sections of the book where technology is particularly appropriate, marginal icons direct students to TEC modules that provide a laboratory environment in which they can explore the topic in different ways and at different levels. **Visuals are animations of figures in text; Modules are more elaborate activities and include exercises.** Instructors can choose to become involved at several different levels, ranging from simply encouraging students to use the Visuals and Modules for independent exploration, to assigning specific exercises from those included with each Module, or to creating additional exercises, labs, and projects that make use of the Visuals and Modules.

HOMEWORK HINTS

Homework Hints presented in the form of questions try to imitate an effective teaching assistant by functioning as a silent tutor. Hints for representative exercises (usually odd-numbered) are included in every section of the text, indicated by printing the exercise number in red. They are constructed so as not to reveal any more of the actual solution than is minimally necessary to make further progress, and are available to students at stewartcalculus.com and in CourseMate and Enhanced WebAssign.

ENHANCED WEBASSIGN

Technology is having an impact on the way homework is assigned to students, particularly in large classes. The use of online homework is growing and its appeal depends on ease of use, grading precision, and reliability. With the seventh edition we have been working with the calculus community and WebAssign to develop a more robust online homework system. Up to 70% of the exercises in each section are assignable as online homework, including free response, multiple choice, and multi-part formats.

The system also includes Active Examples, in which students are guided in step-by-step tutorials through text examples, with links to the textbook and to video solutions. New enhancements to the system include a customizable eBook, a *Show Your Work* feature, *Just in Time* review of precalculus prerequisites, an improved Assignment Editor, and an Answer Evaluator that accepts more mathematically equivalent answers and allows for homework grading in much the same way that an instructor grades.

www.stewartcalculus.com

This site includes the following.

- Homework Hints
- Algebra Review
- Lies My Calculator and Computer Told Me
- History of Mathematics, with links to the better historical websites
- Additional Topics (complete with exercise sets): Fourier Series, Formulas for the Remainder Term in Taylor Series, Rotation of Axes
- Archived Problems (Drill exercises that appeared in previous editions, together with their solutions)
- Challenge Problems (some from the Problems Plus sections from prior editions)
- Links, for particular topics, to outside web resources
- Selected Tools for Enriching Calculus (TEC) Modules and Visuals

Content

- Diagnostic Tests** The book begins with four diagnostic tests, in Basic Algebra, Analytic Geometry, Functions, and Trigonometry.
- A Preview of Calculus** This is an overview of the subject and includes a list of questions to motivate the study of calculus.
- 1 Functions and Models** From the beginning, multiple representations of functions are stressed: verbal, numerical, visual, and algebraic. A discussion of mathematical models leads to a review of the standard functions, including exponential and logarithmic functions, from these four points of view.
- 2 Limits and Derivatives** The material on limits is motivated by a prior discussion of the tangent and velocity problems. Limits are treated from descriptive, graphical, numerical, and algebraic points of view. Section 2.4, on the precise ε - δ definition of a limit, is an optional section. Sections 2.7 and 2.8 deal with derivatives (especially with functions defined graphically and numerically) before the differentiation rules are covered in Chapter 3. Here the examples and exercises explore the meanings of derivatives in various contexts. Higher derivatives are introduced in Section 2.8.
- 3 Differentiation Rules** All the basic functions, including exponential, logarithmic, and inverse trigonometric functions, are differentiated here. When derivatives are computed in applied situations, students are asked to explain their meanings. Exponential growth and decay are covered in this chapter.
- 4 Applications of Differentiation** The basic facts concerning extreme values and shapes of curves are deduced from the Mean Value Theorem. Graphing with technology emphasizes the interaction between calculus and calculators and the analysis of families of curves. Some substantial optimization problems are provided, including an explanation of why you need to raise your head 42° to see the top of a rainbow.
- 5 Integrals** The area problem and the distance problem serve to motivate the definite integral, with sigma notation introduced as needed. (Full coverage of sigma notation is provided in Appendix E.) Emphasis is placed on explaining the meanings of integrals in various contexts and on estimating their values from graphs and tables.
- 6 Applications of Integration** Here I present the applications of integration—area, volume, work, average value—that can reasonably be done without specialized techniques of integration. General methods are emphasized. The goal is for students to be able to divide a quantity into small pieces, estimate with Riemann sums, and recognize the limit as an integral.
- 7 Techniques of Integration** All the standard methods are covered but, of course, the real challenge is to be able to recognize which technique is best used in a given situation. Accordingly, in Section 7.5, I present a strategy for integration. The use of computer algebra systems is discussed in Section 7.6.
- 8 Further Applications of Integration** Here are the applications of integration—arc length and surface area—for which it is useful to have available all the techniques of integration, as well as applications to biology, economics, and physics (hydrostatic force and centers of mass). I have also included a section on probability. There are more applications here than can realistically be covered in a given course. Instructors should select applications suitable for their students and for which they themselves have enthusiasm.

- 9 Differential Equations** Modeling is the theme that unifies this introductory treatment of differential equations. Direction fields and Euler’s method are studied before separable and linear equations are solved explicitly, so that qualitative, numerical, and analytic approaches are given equal consideration. These methods are applied to the exponential, logistic, and other models for population growth. The first four or five sections of this chapter serve as a good introduction to first-order differential equations. An optional final section uses predator-prey models to illustrate systems of differential equations.
- 10 Parametric Equations and Polar Coordinates** This chapter introduces parametric and polar curves and applies the methods of calculus to them. Parametric curves are well suited to laboratory projects; the three presented here involve families of curves and Bézier curves. A brief treatment of conic sections in polar coordinates prepares the way for Kepler’s Laws in Chapter 13.
- 11 Infinite Sequences and Series** The convergence tests have intuitive justifications (see page 714) as well as formal proofs. Numerical estimates of sums of series are based on which test was used to prove convergence. The emphasis is on Taylor series and polynomials and their applications to physics. Error estimates include those from graphing devices.
- 12 Vectors and The Geometry of Space** The material on three-dimensional analytic geometry and vectors is divided into two chapters. Chapter 12 deals with vectors, the dot and cross products, lines, planes, and surfaces.
- 13 Vector Functions** This chapter covers vector-valued functions, their derivatives and integrals, the length and curvature of space curves, and velocity and acceleration along space curves, culminating in Kepler’s laws.
- 14 Partial Derivatives** Functions of two or more variables are studied from verbal, numerical, visual, and algebraic points of view. In particular, I introduce partial derivatives by looking at a specific column in a table of values of the heat index (perceived air temperature) as a function of the actual temperature and the relative humidity.
- 15 Multiple Integrals** Contour maps and the Midpoint Rule are used to estimate the average snowfall and average temperature in given regions. Double and triple integrals are used to compute probabilities, surface areas, and (in projects) volumes of hyperspheres and volumes of intersections of three cylinders. Cylindrical and spherical coordinates are introduced in the context of evaluating triple integrals.
- 16 Vector Calculus** Vector fields are introduced through pictures of velocity fields showing San Francisco Bay wind patterns. The similarities among the Fundamental Theorem for line integrals, Green’s Theorem, Stokes’ Theorem, and the Divergence Theorem are emphasized.
- 17 Second-Order Differential Equations** Since first-order differential equations are covered in Chapter 9, this final chapter deals with second-order linear differential equations, their application to vibrating springs and electric circuits, and series solutions.

Ancillaries

Calculus, Early Transcendentals, Seventh Edition, is supported by a complete set of ancillaries developed under my direction. Each piece has been designed to enhance student understanding and to facilitate creative instruction. With this edition, new media and technologies have been developed that help students to visualize calculus and instructors to customize content to better align with the way they teach their course. The tables on pages xxi–xxii describe each of these ancillaries.

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The preparation of this and previous editions has involved much time spent reading the reasoned (but sometimes contradictory) advice from a large number of astute reviewers. I greatly appreciate the time they spent to understand my motivation for the approach taken. I have learned something from each of them.

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JAMES STEWART

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Ancillaries for Students

Student Solutions Manual

Single Variable Early Transcendentals

By Daniel Anderson, Jeffery A. Cole, and Daniel Drucker
ISBN 0-8400-4934-X

Multivariable

By Dan Clegg and Barbara Frank
ISBN 0-8400-4945-5

Provides completely worked-out solutions to all odd-numbered exercises in the text, giving students a chance to check their answers and ensure they took the correct steps to arrive at an answer.

Study Guide

Single Variable Early Transcendentals

By Richard St. Andre
ISBN 0-8400-5420-3

Multivariable

By Richard St. Andre
ISBN 0-8400-5410-6

For each section of the text, the Study Guide provides students with a brief introduction, a short list of concepts to master, as

well as summary and focus questions with explained answers. The Study Guide also contains "Technology Plus" questions, and multiple-choice "On Your Own" exam-style questions.

CalcLabs with Maple

Single Variable By Philip B. Yasskin and Robert Lopez
ISBN 0-8400-5811-X

Multivariable By Philip B. Yasskin and Robert Lopez
ISBN 0-8400-5812-8

CalcLabs with Mathematica

Single Variable By Selwyn Hollis
ISBN 0-8400-5814-4

Multivariable By Selwyn Hollis
ISBN 0-8400-5813-6

Each of these comprehensive lab manuals will help students learn to use the technology tools available to them. CalcLabs contain clearly explained exercises and a variety of labs and projects to accompany the text.

A Companion to Calculus

By Dennis Ebersole, Doris Schattschneider, Alicia Sevilla, and Kay Somers
ISBN 0-495-01124-X

Written to improve algebra and problem-solving skills of students taking a Calculus course, every chapter in this companion is keyed to a calculus topic, providing conceptual background and specific algebra techniques needed to understand and solve calculus problems related to that topic. It is designed for calculus courses that integrate the review of precalculus concepts or for individual use.

Linear Algebra for Calculus

by Konrad J. Heuvers, William P. Francis, John H. Kuisti, Deborah F. Lockhart, Daniel S. Moak, and Gene M. Ortner
ISBN 0-534-25248-6

This comprehensive book, designed to supplement the calculus course, provides an introduction to and review of the basic ideas of linear algebra.





To the Student

Reading a calculus textbook is different from reading a newspaper or a novel, or even a physics book. Don't be discouraged if you have to read a passage more than once in order to understand it. You should have pencil and paper and calculator at hand to sketch a diagram or make a calculation.


Some students start by trying their homework problems and read the text only if they get stuck on an exercise. I suggest that a far better plan is to read and understand a section of the text before attempting the exercises. In particular, you should look at the definitions to see the exact meanings of the terms. And before you read each example, I suggest that you cover up the solution and try solving the problem yourself. You'll get a lot more from looking at the solution if you do so.


Part of the aim of this course is to train you to think logically. Learn to write the solutions of the exercises in a connected, step-by-step fashion with explanatory sentences — not just a string of disconnected equations or formulas.

The answers to the odd-numbered exercises appear at the back of the book, in Appendix I. Some exercises ask for a verbal explanation or interpretation or description. In such cases there is no single correct way of expressing the answer, so don't worry that you haven't found the definitive answer. In addition, there are often several different forms in which to express a numerical or algebraic answer, so if your answer differs from mine, don't immediately assume you're wrong. For example, if the answer given in the back of the book is $\sqrt{2} - 1$ and you obtain $1/(1 + \sqrt{2})$, then you're right and rationalizing the denominator will show that the answers are equivalent.

The icon  indicates an exercise that definitely requires the use of either a graphing calculator or a computer with graphing software. (Section 1.4 discusses the use of these graphing devices and some of the pitfalls that you may encounter.) But that doesn't mean that graphing devices can't be used to check your work on the other exercises as well. The symbol  is

reserved for problems in which the full resources of a computer algebra system (like Derive, Maple, Mathematica, or the TI-89/92) are required.

You will also encounter the symbol , which warns you against committing an error. I have placed this symbol in the margin in situations where I have observed that a large proportion of my students tend to make the same mistake.

Tools for Enriching Calculus, which is a companion to this text, is referred to by means of the symbol  and can be accessed in Enhanced WebAssign and CourseMate (selected Visuals and Modules are available at www.stewartcalculus.com). It directs you to modules in which you can explore aspects of calculus for which the computer is particularly useful.

Homework Hints for representative exercises are indicated by printing the exercise number in red: **5**. These hints can be found on stewartcalculus.com as well as Enhanced WebAssign and CourseMate. The homework hints ask you questions that allow you to make progress toward a solution without actually giving you the answer. You need to pursue each hint in an active manner with pencil and paper to work out the details. If a particular hint doesn't enable you to solve the problem, you can click to reveal the next hint.

I recommend that you keep this book for reference purposes after you finish the course. Because you will likely forget some of the specific details of calculus, the book will serve as a useful reminder when you need to use calculus in subsequent courses. And, because this book contains more material than can be covered in any one course, it can also serve as a valuable resource for a working scientist or engineer.

Calculus is an exciting subject, justly considered to be one of the greatest achievements of the human intellect. I hope you will discover that it is not only useful but also intrinsically beautiful.

JAMES STEWART

Diagnostic Tests

Success in calculus depends to a large extent on knowledge of the mathematics that precedes calculus: algebra, analytic geometry, functions, and trigonometry. The following tests are intended to diagnose weaknesses that you might have in these areas. After taking each test you can check your answers against the given answers and, if necessary, refresh your skills by referring to the review materials that are provided.

A Diagnostic Test: Algebra

1. Evaluate each expression without using a calculator.

- (a) $(-3)^4$ (b) -3^4 (c) 3^{-4}
(d) $\frac{5^{23}}{5^{21}}$ (e) $\left(\frac{2}{3}\right)^{-2}$ (f) $16^{-3/4}$

2. Simplify each expression. Write your answer without negative exponents.

- (a) $\sqrt{200} - \sqrt{32}$
(b) $(3a^3b^3)(4ab^2)^2$
(c) $\left(\frac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2}$

3. Expand and simplify.

- (a) $3(x + 6) + 4(2x - 5)$ (b) $(x + 3)(4x - 5)$
(c) $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$ (d) $(2x + 3)^2$
(e) $(x + 2)^3$

4. Factor each expression.

- (a) $4x^2 - 25$ (b) $2x^2 + 5x - 12$
(c) $x^3 - 3x^2 - 4x + 12$ (d) $x^4 + 27x$
(e) $3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$ (f) $x^3y - 4xy$

5. Simplify the rational expression.

- (a) $\frac{x^2 + 3x + 2}{x^2 - x - 2}$ (b) $\frac{2x^2 - x - 1}{x^2 - 9} \cdot \frac{x + 3}{2x + 1}$
(c) $\frac{x^2}{x^2 - 4} - \frac{x + 1}{x + 2}$ (d) $\frac{\frac{y}{x} - \frac{x}{y}}{\frac{1}{y} - \frac{1}{x}}$

6. Rationalize the expression and simplify.

(a) $\frac{\sqrt{10}}{\sqrt{5} - 2}$

(b) $\frac{\sqrt{4+h} - 2}{h}$

7. Rewrite by completing the square.

(a) $x^2 + x + 1$

(b) $2x^2 - 12x + 11$

8. Solve the equation. (Find only the real solutions.)

(a) $x + 5 = 14 - \frac{1}{2}x$

(b) $\frac{2x}{x+1} = \frac{2x-1}{x}$

(c) $x^2 - x - 12 = 0$

(d) $2x^2 + 4x + 1 = 0$

(e) $x^4 - 3x^2 + 2 = 0$

(f) $3|x-4| = 10$

(g) $2x(4-x)^{-1/2} - 3\sqrt{4-x} = 0$

9. Solve each inequality. Write your answer using interval notation.

(a) $-4 < 5 - 3x \leq 17$

(b) $x^2 < 2x + 8$

(c) $x(x-1)(x+2) > 0$

(d) $|x-4| < 3$

(e) $\frac{2x-3}{x+1} \leq 1$

10. State whether each equation is true or false.

(a) $(p+q)^2 = p^2 + q^2$

(b) $\sqrt{ab} = \sqrt{a}\sqrt{b}$

(c) $\sqrt{a^2 + b^2} = a + b$

(d) $\frac{1+TC}{C} = 1 + T$

(e) $\frac{1}{x-y} = \frac{1}{x} - \frac{1}{y}$

(f) $\frac{1/x}{a/x - b/x} = \frac{1}{a-b}$

Answers to Diagnostic Test A: Algebra

1. (a) 81 (b) -81 (c) $\frac{1}{81}$
(d) 25 (e) $\frac{9}{4}$ (f) $\frac{1}{8}$

2. (a) $6\sqrt{2}$ (b) $48a^5b^7$ (c) $\frac{x}{9y^7}$

3. (a) $11x - 2$ (b) $4x^2 + 7x - 15$
(c) $a - b$ (d) $4x^2 + 12x + 9$
(e) $x^3 + 6x^2 + 12x + 8$

4. (a) $(2x-5)(2x+5)$ (b) $(2x-3)(x+4)$
(c) $(x-3)(x-2)(x+2)$ (d) $x(x+3)(x^2-3x+9)$
(e) $3x^{-1/2}(x-1)(x-2)$ (f) $xy(x-2)(x+2)$

5. (a) $\frac{x+2}{x-2}$ (b) $\frac{x-1}{x-3}$
(c) $\frac{1}{x-2}$ (d) $-(x+y)$

6. (a) $5\sqrt{2} + 2\sqrt{10}$ (b) $\frac{1}{\sqrt{4+h}+2}$

7. (a) $(x + \frac{1}{2})^2 + \frac{3}{4}$ (b) $2(x-3)^2 - 7$

8. (a) 6 (b) 1 (c) -3, 4
(d) $-1 \pm \frac{1}{2}\sqrt{2}$ (e) $\pm 1, \pm\sqrt{2}$ (f) $\frac{2}{3}, \frac{22}{3}$
(g) $\frac{12}{5}$

9. (a) $[-4, 3)$ (b) $(-2, 4)$
(c) $(-2, 0) \cup (1, \infty)$ (d) $(1, 7)$
(e) $(-1, 4]$

10. (a) False (b) True (c) False
(d) False (e) False (f) True

If you have had difficulty with these problems, you may wish to consult the Review of Algebra on the website www.stewartcalculus.com

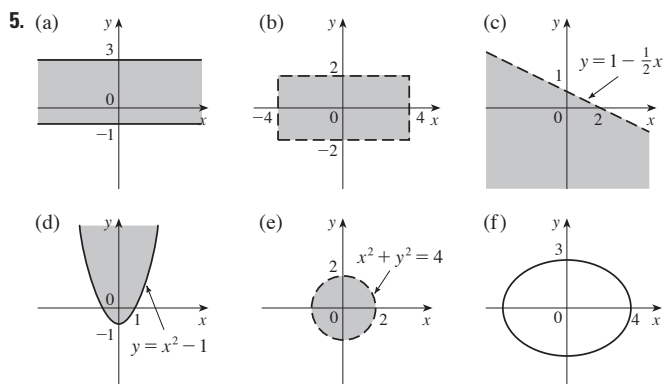
B Diagnostic Test: Analytic Geometry

- Find an equation for the line that passes through the point $(2, -5)$ and
 - has slope -3
 - is parallel to the x -axis
 - is parallel to the y -axis
 - is parallel to the line $2x - 4y = 3$
- Find an equation for the circle that has center $(-1, 4)$ and passes through the point $(3, -2)$.
- Find the center and radius of the circle with equation $x^2 + y^2 - 6x + 10y + 9 = 0$.
- Let $A(-7, 4)$ and $B(5, -12)$ be points in the plane.
 - Find the slope of the line that contains A and B .
 - Find an equation of the line that passes through A and B . What are the intercepts?
 - Find the midpoint of the segment AB .
 - Find the length of the segment AB .
 - Find an equation of the perpendicular bisector of AB .
 - Find an equation of the circle for which AB is a diameter.
- Sketch the region in the xy -plane defined by the equation or inequalities.

(a) $-1 \leq y \leq 3$	(b) $ x < 4$ and $ y < 2$
(c) $y < 1 - \frac{1}{2}x$	(d) $y \geq x^2 - 1$
(e) $x^2 + y^2 < 4$	(f) $9x^2 + 16y^2 = 144$

Answers to Diagnostic Test B: Analytic Geometry

- (a) $y = -3x + 1$
(c) $x = 2$
- $(x + 1)^2 + (y - 4)^2 = 52$
- Center $(3, -5)$, radius 5
- (a) $-\frac{4}{3}$
(b) $4x + 3y + 16 = 0$; x -intercept -4 , y -intercept $-\frac{16}{3}$
(c) $(-1, -4)$
(d) 20
(e) $3x - 4y = 13$
(f) $(x + 1)^2 + (y + 4)^2 = 100$



If you have had difficulty with these problems, you may wish to consult the review of analytic geometry in Appendixes B and C.

C Diagnostic Test: Functions

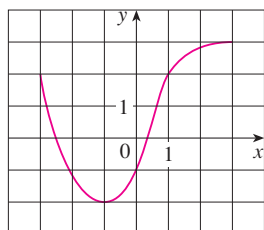


FIGURE FOR PROBLEM 1

- The graph of a function f is given at the left.
 - State the value of $f(-1)$.
 - Estimate the value of $f(2)$.
 - For what values of x is $f(x) = 2$?
 - Estimate the values of x such that $f(x) = 0$.
 - State the domain and range of f .
- If $f(x) = x^3$, evaluate the difference quotient $\frac{f(2+h) - f(2)}{h}$ and simplify your answer.
- Find the domain of the function.
 - $f(x) = \frac{2x+1}{x^2+x-2}$
 - $g(x) = \frac{\sqrt[3]{x}}{x^2+1}$
 - $h(x) = \sqrt{4-x} + \sqrt{x^2-1}$
- How are graphs of the functions obtained from the graph of f ?
 - $y = -f(x)$
 - $y = 2f(x) - 1$
 - $y = f(x-3) + 2$
- Without using a calculator, make a rough sketch of the graph.
 - $y = x^3$
 - $y = (x+1)^3$
 - $y = (x-2)^3 + 3$
 - $y = 4 - x^2$
 - $y = \sqrt{x}$
 - $y = 2\sqrt{x}$
 - $y = -2^x$
 - $y = 1 + x^{-1}$
- Let $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leq 0 \\ 2x + 1 & \text{if } x > 0 \end{cases}$
 - Evaluate $f(-2)$ and $f(1)$.
 - Sketch the graph of f .
- If $f(x) = x^2 + 2x - 1$ and $g(x) = 2x - 3$, find each of the following functions.
 - $f \circ g$
 - $g \circ f$
 - $g \circ g \circ g$

Answers to Diagnostic Test C: Functions

- (a) -2
 - (b) 2.8
 - (c) $-3, 1$
 - (d) $-2.5, 0.3$
 - (e) $[-3, 3], [-2, 3]$
- $12 + 6h + h^2$
- (a) $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$
 - (b) $(-\infty, \infty)$
 - (c) $(-\infty, -1] \cup [1, 4]$
- (a) Reflect about the x -axis
 - (b) Stretch vertically by a factor of 2, then shift 1 unit downward
 - (c) Shift 3 units to the right and 2 units upward
- (a)
 - (b)
 - (c)
- (a) $-3, 3$
 - (b)
- (a)
 - (b)
 - (c)
- (a) $(f \circ g)(x) = 4x^2 - 8x + 2$
 - (b) $(g \circ f)(x) = 2x^2 + 4x - 5$
 - (c) $(g \circ g \circ g)(x) = 8x - 21$

If you have had difficulty with these problems, you should look at Sections 1.1–1.3 of this book.

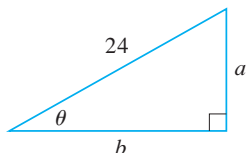
D Diagnostic Test: Trigonometry

FIGURE FOR PROBLEM 5

- Convert from degrees to radians.
(a) 300° (b) -18°
- Convert from radians to degrees.
(a) $5\pi/6$ (b) 2
- Find the length of an arc of a circle with radius 12 cm if the arc subtends a central angle of 30° .
- Find the exact values.
(a) $\tan(\pi/3)$ (b) $\sin(7\pi/6)$ (c) $\sec(5\pi/3)$
- Express the lengths a and b in the figure in terms of θ .
- If $\sin x = \frac{1}{3}$ and $\sec y = \frac{5}{4}$, where x and y lie between 0 and $\pi/2$, evaluate $\sin(x + y)$.
- Prove the identities.
(a) $\tan \theta \sin \theta + \cos \theta = \sec \theta$
(b) $\frac{2 \tan x}{1 + \tan^2 x} = \sin 2x$
- Find all values of x such that $\sin 2x = \sin x$ and $0 \leq x \leq 2\pi$.
- Sketch the graph of the function $y = 1 + \sin 2x$ without using a calculator.

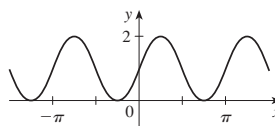
Answers to Diagnostic Test D: Trigonometry

- (a) $5\pi/3$ (b) $-\pi/10$
- (a) 150° (b) $360^\circ/\pi \approx 114.6^\circ$
- 2π cm
- (a) $\sqrt{3}$ (b) $-\frac{1}{2}$ (c) 2
- (a) $24 \sin \theta$ (b) $24 \cos \theta$

6. $\frac{1}{15}(4 + 6\sqrt{2})$

8. $0, \pi/3, \pi, 5\pi/3, 2\pi$

9.



If you have had difficulty with these problems, you should look at Appendix D of this book.

A Preview of Calculus



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By the time you finish this course, you will be able to estimate the number of laborers needed to build a pyramid, explain the formation and location of rainbows, design a roller coaster for a smooth ride, and calculate the force on a dam.



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Calculus is fundamentally different from the mathematics that you have studied previously: calculus is less static and more dynamic. It is concerned with change and motion; it deals with quantities that approach other quantities. For that reason it may be useful to have an overview of the subject before beginning its intensive study. Here we give a glimpse of some of the main ideas of calculus by showing how the concept of a limit arises when we attempt to solve a variety of problems.

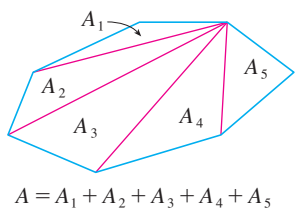


FIGURE 1

The Area Problem

The origins of calculus go back at least 2500 years to the ancient Greeks, who found areas using the “method of exhaustion.” They knew how to find the area A of any polygon by dividing it into triangles as in Figure 1 and adding the areas of these triangles.

It is a much more difficult problem to find the area of a curved figure. The Greek method of exhaustion was to inscribe polygons in the figure and circumscribe polygons about the figure and then let the number of sides of the polygons increase. Figure 2 illustrates this process for the special case of a circle with inscribed regular polygons.

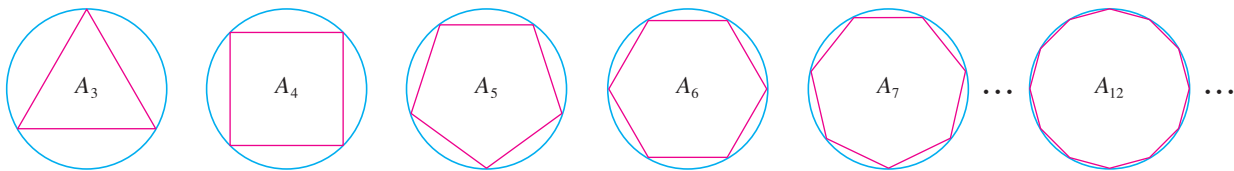


FIGURE 2

Let A_n be the area of the inscribed polygon with n sides. As n increases, it appears that A_n becomes closer and closer to the area of the circle. We say that the area of the circle is the *limit* of the areas of the inscribed polygons, and we write

$$A = \lim_{n \rightarrow \infty} A_n$$

TEC In the Preview Visual, you can see how areas of inscribed and circumscribed polygons approximate the area of a circle.

The Greeks themselves did not use limits explicitly. However, by indirect reasoning, Eudoxus (fifth century BC) used exhaustion to prove the familiar formula for the area of a circle: $A = \pi r^2$.

We will use a similar idea in Chapter 5 to find areas of regions of the type shown in Figure 3. We will approximate the desired area A by areas of rectangles (as in Figure 4), let the width of the rectangles decrease, and then calculate A as the limit of these sums of areas of rectangles.

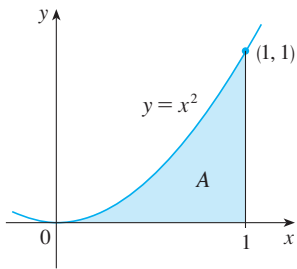


FIGURE 3

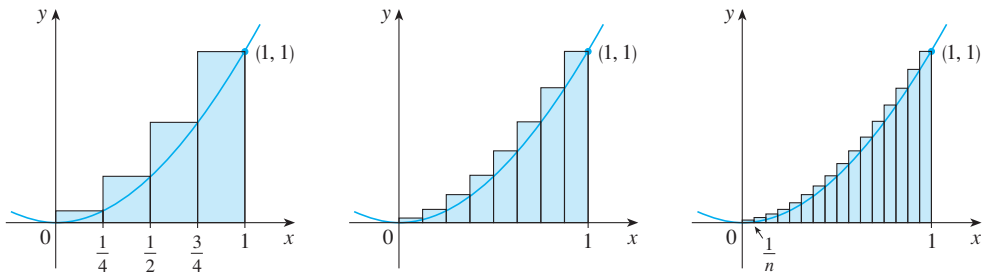


FIGURE 4

The area problem is the central problem in the branch of calculus called *integral calculus*. The techniques that we will develop in Chapter 5 for finding areas will also enable us to compute the volume of a solid, the length of a curve, the force of water against a dam, the mass and center of gravity of a rod, and the work done in pumping water out of a tank.

The Tangent Problem

Consider the problem of trying to find an equation of the tangent line t to a curve with equation $y = f(x)$ at a given point P . (We will give a precise definition of a tangent line in

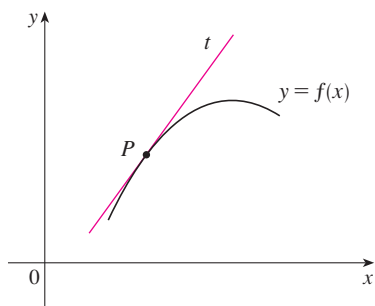


FIGURE 5
The tangent line at P

Chapter 2. For now you can think of it as a line that touches the curve at P as in Figure 5.) Since we know that the point P lies on the tangent line, we can find the equation of t if we know its slope m . The problem is that we need two points to compute the slope and we know only one point, P , on t . To get around the problem we first find an approximation to m by taking a nearby point Q on the curve and computing the slope m_{PQ} of the secant line PQ . From Figure 6 we see that

1

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Now imagine that Q moves along the curve toward P as in Figure 7. You can see that the secant line rotates and approaches the tangent line as its limiting position. This means that the slope m_{PQ} of the secant line becomes closer and closer to the slope m of the tangent line. We write

$$m = \lim_{Q \rightarrow P} m_{PQ}$$

and we say that m is the limit of m_{PQ} as Q approaches P along the curve. Since x approaches a as Q approaches P , we could also use Equation 1 to write

2

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

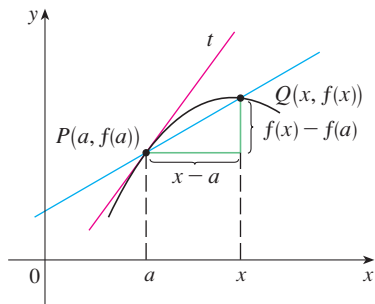


FIGURE 6
The secant line PQ

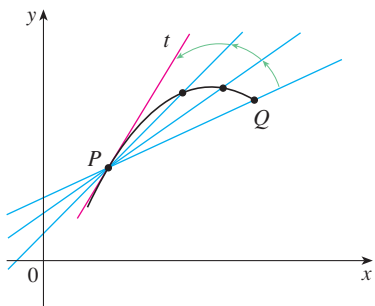


FIGURE 7
Secant lines approaching the tangent line

Specific examples of this procedure will be given in Chapter 2.

The tangent problem has given rise to the branch of calculus called *differential calculus*, which was not invented until more than 2000 years after integral calculus. The main ideas behind differential calculus are due to the French mathematician Pierre Fermat (1601–1665) and were developed by the English mathematicians John Wallis (1616–1703), Isaac Barrow (1630–1677), and Isaac Newton (1642–1727) and the German mathematician Gottfried Leibniz (1646–1716).

The two branches of calculus and their chief problems, the area problem and the tangent problem, appear to be very different, but it turns out that there is a very close connection between them. The tangent problem and the area problem are inverse problems in a sense that will be described in Chapter 5.

Velocity

When we look at the speedometer of a car and read that the car is traveling at 48 mi/h, what does that information indicate to us? We know that if the velocity remains constant, then after an hour we will have traveled 48 mi. But if the velocity of the car varies, what does it mean to say that the velocity at a given instant is 48 mi/h?

In order to analyze this question, let's examine the motion of a car that travels along a straight road and assume that we can measure the distance traveled by the car (in feet) at 1-second intervals as in the following chart:

t = Time elapsed (s)	0	1	2	3	4	5
d = Distance (ft)	0	2	9	24	42	71

As a first step toward finding the velocity after 2 seconds have elapsed, we find the average velocity during the time interval $2 \leq t \leq 4$:

$$\begin{aligned}\text{average velocity} &= \frac{\text{change in position}}{\text{time elapsed}} \\ &= \frac{42 - 9}{4 - 2} \\ &= 16.5 \text{ ft/s}\end{aligned}$$

Similarly, the average velocity in the time interval $2 \leq t \leq 3$ is

$$\text{average velocity} = \frac{24 - 9}{3 - 2} = 15 \text{ ft/s}$$

We have the feeling that the velocity at the instant $t = 2$ can't be much different from the average velocity during a short time interval starting at $t = 2$. So let's imagine that the distance traveled has been measured at 0.1-second time intervals as in the following chart:

t	2.0	2.1	2.2	2.3	2.4	2.5
d	9.00	10.02	11.16	12.45	13.96	15.80

Then we can compute, for instance, the average velocity over the time interval $[2, 2.5]$:

$$\text{average velocity} = \frac{15.80 - 9.00}{2.5 - 2} = 13.6 \text{ ft/s}$$

The results of such calculations are shown in the following chart:

Time interval	$[2, 3]$	$[2, 2.5]$	$[2, 2.4]$	$[2, 2.3]$	$[2, 2.2]$	$[2, 2.1]$
Average velocity (ft/s)	15.0	13.6	12.4	11.5	10.8	10.2

The average velocities over successively smaller intervals appear to be getting closer to a number near 10, and so we expect that the velocity at exactly $t = 2$ is about 10 ft/s. In Chapter 2 we will define the instantaneous velocity of a moving object as the limiting value of the average velocities over smaller and smaller time intervals.

In Figure 8 we show a graphical representation of the motion of the car by plotting the distance traveled as a function of time. If we write $d = f(t)$, then $f(t)$ is the number of feet traveled after t seconds. The average velocity in the time interval $[2, t]$ is

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{f(t) - f(2)}{t - 2}$$

which is the same as the slope of the secant line PQ in Figure 8. The velocity v when $t = 2$ is the limiting value of this average velocity as t approaches 2; that is,

$$v = \lim_{t \rightarrow 2} \frac{f(t) - f(2)}{t - 2}$$

and we recognize from Equation 2 that this is the same as the slope of the tangent line to the curve at P .

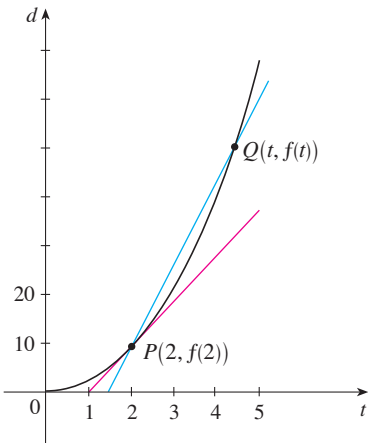


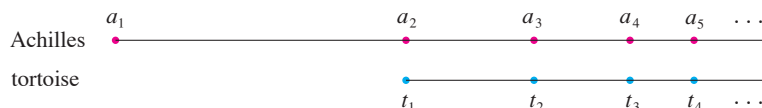
FIGURE 8

Thus, when we solve the tangent problem in differential calculus, we are also solving problems concerning velocities. The same techniques also enable us to solve problems involving rates of change in all of the natural and social sciences.

The Limit of a Sequence

In the fifth century BC the Greek philosopher Zeno of Elea posed four problems, now known as *Zeno's paradoxes*, that were intended to challenge some of the ideas concerning space and time that were held in his day. Zeno's second paradox concerns a race between the Greek hero Achilles and a tortoise that has been given a head start. Zeno argued, as follows, that Achilles could never pass the tortoise: Suppose that Achilles starts at position a_1 and the tortoise starts at position t_1 . (See Figure 9.) When Achilles reaches the point $a_2 = t_1$, the tortoise is farther ahead at position t_2 . When Achilles reaches $a_3 = t_2$, the tortoise is at t_3 . This process continues indefinitely and so it appears that the tortoise will always be ahead! But this defies common sense.

FIGURE 9



One way of explaining this paradox is with the idea of a *sequence*. The successive positions of Achilles (a_1, a_2, a_3, \dots) or the successive positions of the tortoise (t_1, t_2, t_3, \dots) form what is known as a sequence.

In general, a sequence $\{a_n\}$ is a set of numbers written in a definite order. For instance, the sequence

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right\}$$

can be described by giving the following formula for the n th term:

$$a_n = \frac{1}{n}$$

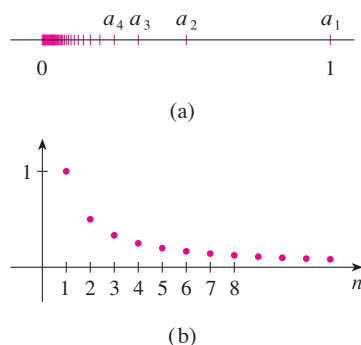


FIGURE 10

We can visualize this sequence by plotting its terms on a number line as in Figure 10(a) or by drawing its graph as in Figure 10(b). Observe from either picture that the terms of the sequence $a_n = 1/n$ are becoming closer and closer to 0 as n increases. In fact, we can find terms as small as we please by making n large enough. We say that the limit of the sequence is 0, and we indicate this by writing

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

In general, the notation

$$\lim_{n \rightarrow \infty} a_n = L$$

is used if the terms a_n approach the number L as n becomes large. This means that the numbers a_n can be made as close as we like to the number L by taking n sufficiently large.

The concept of the limit of a sequence occurs whenever we use the decimal representation of a real number. For instance, if

$$a_1 = 3.1$$

$$a_2 = 3.14$$

$$a_3 = 3.141$$

$$a_4 = 3.1415$$

$$a_5 = 3.14159$$

$$a_6 = 3.141592$$

$$a_7 = 3.1415926$$

$$\vdots$$

then

$$\lim_{n \rightarrow \infty} a_n = \pi$$

The terms in this sequence are rational approximations to π .

Let's return to Zeno's paradox. The successive positions of Achilles and the tortoise form sequences $\{a_n\}$ and $\{t_n\}$, where $a_n < t_n$ for all n . It can be shown that both sequences have the same limit:

$$\lim_{n \rightarrow \infty} a_n = p = \lim_{n \rightarrow \infty} t_n$$

It is precisely at this point p that Achilles overtakes the tortoise.

The Sum of a Series

Another of Zeno's paradoxes, as passed on to us by Aristotle, is the following: "A man standing in a room cannot walk to the wall. In order to do so, he would first have to go half the distance, then half the remaining distance, and then again half of what still remains. This process can always be continued and can never be ended." (See Figure 11.)

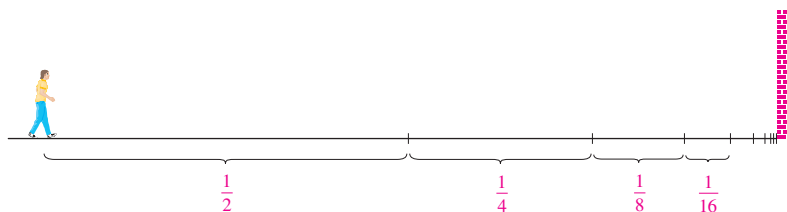


FIGURE 11

Of course, we know that the man can actually reach the wall, so this suggests that perhaps the total distance can be expressed as the sum of infinitely many smaller distances as follows:

$$\boxed{3} \quad 1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$$

Zeno was arguing that it doesn't make sense to add infinitely many numbers together. But there are other situations in which we implicitly use infinite sums. For instance, in decimal notation, the symbol $0.\overline{3} = 0.3333 \dots$ means

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000} + \dots$$

and so, in some sense, it must be true that

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000} + \dots = \frac{1}{3}$$

More generally, if d_n denotes the n th digit in the decimal representation of a number, then

$$0.d_1d_2d_3d_4\dots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \dots + \frac{d_n}{10^n} + \dots$$

Therefore some infinite sums, or infinite series as they are called, have a meaning. But we must define carefully what the sum of an infinite series is.

Returning to the series in Equation 3, we denote by s_n the sum of the first n terms of the series. Thus

$$s_1 = \frac{1}{2} = 0.5$$

$$s_2 = \frac{1}{2} + \frac{1}{4} = 0.75$$

$$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.875$$

$$s_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 0.9375$$

$$s_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = 0.96875$$

$$s_6 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = 0.984375$$

$$s_7 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} = 0.9921875$$

$$\vdots$$

$$s_{10} = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{1024} \approx 0.99902344$$

$$\vdots$$

$$s_{16} = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{16}} \approx 0.99998474$$

Observe that as we add more and more terms, the partial sums become closer and closer to 1. In fact, it can be shown that by taking n large enough (that is, by adding sufficiently many terms of the series), we can make the partial sum s_n as close as we please to the number 1. It therefore seems reasonable to say that the sum of the infinite series is 1 and to write

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots = 1$$

In other words, the reason the sum of the series is 1 is that

$$\lim_{n \rightarrow \infty} s_n = 1$$

In Chapter 11 we will discuss these ideas further. We will then use Newton's idea of combining infinite series with differential and integral calculus.

Summary

We have seen that the concept of a limit arises in trying to find the area of a region, the slope of a tangent to a curve, the velocity of a car, or the sum of an infinite series. In each case the common theme is the calculation of a quantity as the limit of other, easily calculated quantities. It is this basic idea of a limit that sets calculus apart from other areas of mathematics. In fact, we could define calculus as the part of mathematics that deals with limits.

After Sir Isaac Newton invented his version of calculus, he used it to explain the motion of the planets around the sun. Today calculus is used in calculating the orbits of satellites and spacecraft, in predicting population sizes, in estimating how fast oil prices rise or fall, in forecasting weather, in measuring the cardiac output of the heart, in calculating life insurance premiums, and in a great variety of other areas. We will explore some of these uses of calculus in this book.

In order to convey a sense of the power of the subject, we end this preview with a list of some of the questions that you will be able to answer using calculus:

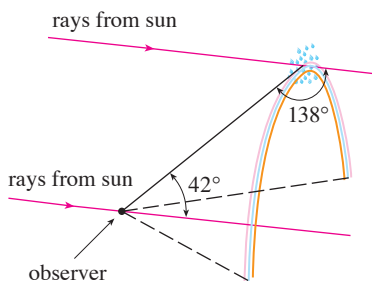


FIGURE 12

1. How can we explain the fact, illustrated in Figure 12, that the angle of elevation from an observer up to the highest point in a rainbow is 42° ? (See page 282.)
2. How can we explain the shapes of cans on supermarket shelves? (See page 337.)
3. Where is the best place to sit in a movie theater? (See page 456.)
4. How can we design a roller coaster for a smooth ride? (See page 184.)
5. How far away from an airport should a pilot start descent? (See page 208.)
6. How can we fit curves together to design shapes to represent letters on a laser printer? (See page 653.)
7. How can we estimate the number of workers that were needed to build the Great Pyramid of Khufu in ancient Egypt? (See page 451.)
8. Where should an infielder position himself to catch a baseball thrown by an outfielder and relay it to home plate? (See page 456.)
9. Does a ball thrown upward take longer to reach its maximum height or to fall back to its original height? (See page 604.)
10. How can we explain the fact that planets and satellites move in elliptical orbits? (See page 868.)
11. How can we distribute water flow among turbines at a hydroelectric station so as to maximize the total energy production? (See page 966.)
12. If a marble, a squash ball, a steel bar, and a lead pipe roll down a slope, which of them reaches the bottom first? (See page 1039.)

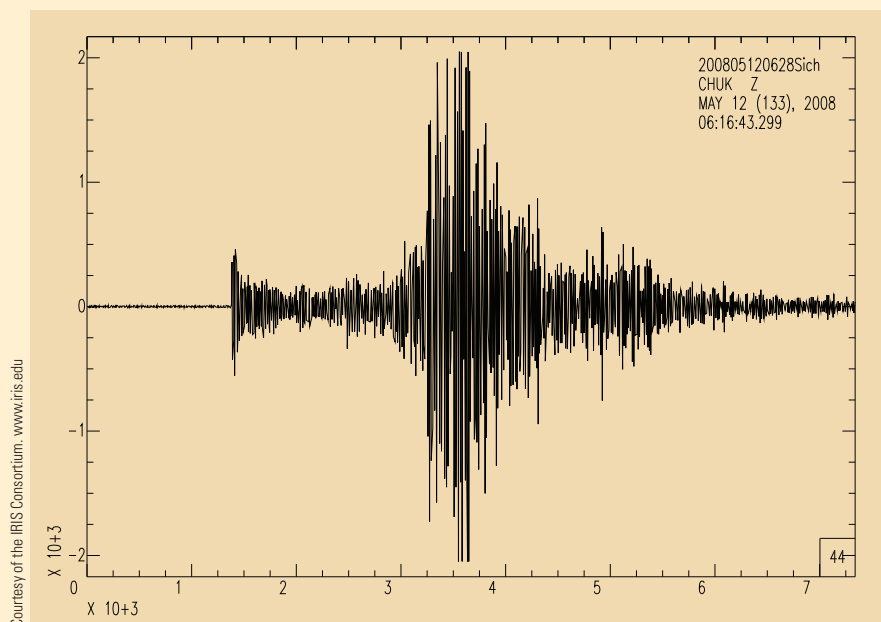
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Functions and Models



Often a graph is the best way to represent a function because it conveys so much information at a glance. Shown is a graph of the ground acceleration created by the 2008 earthquake in Sichuan province in China. The hardest hit town was Beichuan, as pictured.

© Mark Ralston / AFP / Getty Images



The fundamental objects that we deal with in calculus are functions. This chapter prepares the way for calculus by discussing the basic ideas concerning functions, their graphs, and ways of transforming and combining them. We stress that a function can be represented in different ways: by an equation, in a table, by a graph, or in words. We look at the main types of functions that occur in calculus and describe the process of using these functions as mathematical models of real-world phenomena. We also discuss the use of graphing calculators and graphing software for computers.

1.1 Four Ways to Represent a Function

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080
2010	6870

Functions arise whenever one quantity depends on another. Consider the following four situations.

- A. The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A , and we say that A is a *function* of r .
- B. The human population of the world P depends on the time t . The table gives estimates of the world population $P(t)$ at time t , for certain years. For instance,

$$P(1950) \approx 2,560,000,000$$

But for each value of the time t there is a corresponding value of P , and we say that P is a function of t .

- C. The cost C of mailing an envelope depends on its weight w . Although there is no simple formula that connects w and C , the post office has a rule for determining C when w is known.
- D. The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t . Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of t , the graph provides a corresponding value of a .

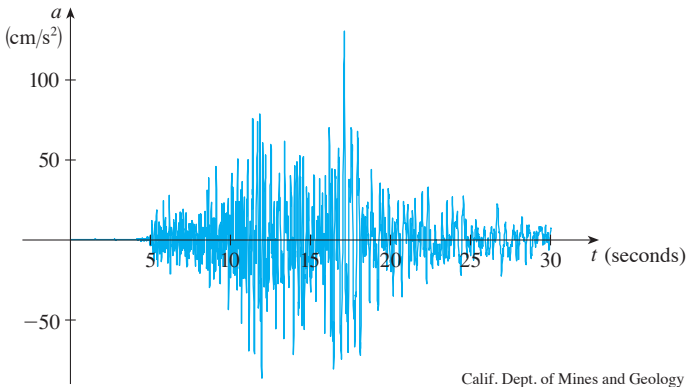


FIGURE 1
Vertical ground acceleration during
the Northridge earthquake

Each of these examples describes a rule whereby, given a number (r , t , w , or t), another number (A , P , C , or a) is assigned. In each case we say that the second number is a function of the first number.

A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E .

We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the **domain** of the function. The number $f(x)$ is the **value of f at x** and is read “ f of x .” The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain. A symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**. A symbol that represents a number in the *range* of f is called a **dependent variable**. In Example A, for instance, r is the independent variable and A is the dependent variable.



FIGURE 2
Machine diagram for a function f

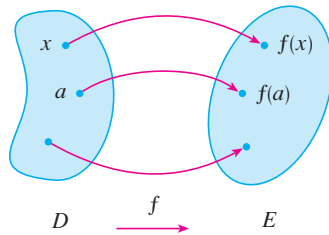


FIGURE 3
Arrow diagram for f

It's helpful to think of a function as a **machine** (see Figure 2). If x is in the domain of the function f , then when x enters the machine, it's accepted as an input and the machine produces an output $f(x)$ according to the rule of the function. Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

The preprogrammed functions in a calculator are good examples of a function as a machine. For example, the square root key on your calculator computes such a function. You press the key labeled $\sqrt{}$ (or \sqrt{x}) and enter the input x . If $x < 0$, then x is not in the domain of this function; that is, x is not an acceptable input, and the calculator will indicate an error. If $x \geq 0$, then an *approximation* to \sqrt{x} will appear in the display. Thus the \sqrt{x} key on your calculator is not quite the same as the exact mathematical function f defined by $f(x) = \sqrt{x}$.

Another way to picture a function is by an **arrow diagram** as in Figure 3. Each arrow connects an element of D to an element of E . The arrow indicates that $f(x)$ is associated with x , $f(a)$ is associated with a , and so on.

The most common method for visualizing a function is its graph. If f is a function with domain D , then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

(Notice that these are input-output pairs.) In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f .

The graph of a function f gives us a useful picture of the behavior or “life history” of a function. Since the y -coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x (see Figure 4). The graph of f also allows us to picture the domain of f on the x -axis and its range on the y -axis as in Figure 5.

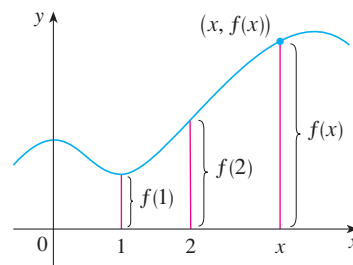


FIGURE 4

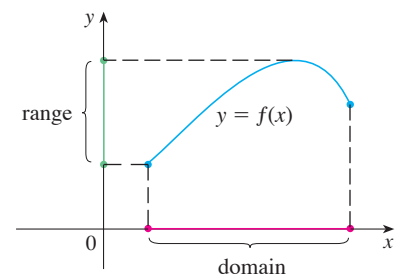


FIGURE 5

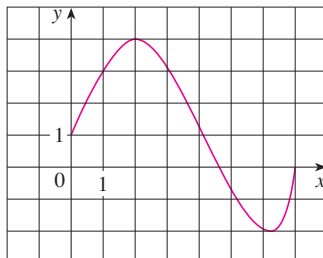


FIGURE 6

EXAMPLE 1 The graph of a function f is shown in Figure 6.

- Find the values of $f(1)$ and $f(5)$.
- What are the domain and range of f ?

SOLUTION

(a) We see from Figure 6 that the point $(1, 3)$ lies on the graph of f , so the value of f at 1 is $f(1) = 3$. (In other words, the point on the graph that lies above $x = 1$ is 3 units above the x -axis.)

When $x = 5$, the graph lies about 0.7 unit below the x -axis, so we estimate that $f(5) \approx -0.7$.

(b) We see that $f(x)$ is defined when $0 \leq x \leq 7$, so the domain of f is the closed interval $[0, 7]$. Notice that f takes on all values from -2 to 4 , so the range of f is

$$\{y \mid -2 \leq y \leq 4\} = [-2, 4]$$

The notation for intervals is given in Appendix A.

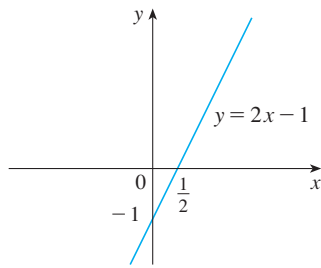


FIGURE 7

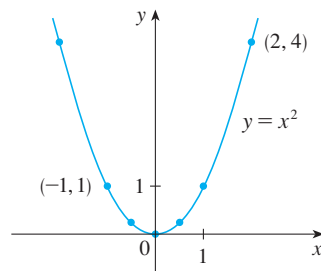


FIGURE 8

EXAMPLE 2 Sketch the graph and find the domain and range of each function.

(a) $f(x) = 2x - 1$

(b) $g(x) = x^2$

SOLUTION

(a) The equation of the graph is $y = 2x - 1$, and we recognize this as being the equation of a line with slope 2 and y-intercept -1 . (Recall the slope-intercept form of the equation of a line: $y = mx + b$. See Appendix B.) This enables us to sketch a portion of the graph of f in Figure 7. The expression $2x - 1$ is defined for all real numbers, so the domain of f is the set of all real numbers, which we denote by \mathbb{R} . The graph shows that the range is also \mathbb{R} .

(b) Since $g(2) = 2^2 = 4$ and $g(-1) = (-1)^2 = 1$, we could plot the points $(2, 4)$ and $(-1, 1)$, together with a few other points on the graph, and join them to produce the graph (Figure 8). The equation of the graph is $y = x^2$, which represents a parabola (see Appendix C). The domain of g is \mathbb{R} . The range of g consists of all values of $g(x)$, that is, all numbers of the form x^2 . But $x^2 \geq 0$ for all numbers x and any positive number y is a square. So the range of g is $\{y \mid y \geq 0\} = [0, \infty)$. This can also be seen from Figure 8.

EXAMPLE 3 If $f(x) = 2x^2 - 5x + 1$ and $h \neq 0$, evaluate $\frac{f(a+h) - f(a)}{h}$.

SOLUTION We first evaluate $f(a+h)$ by replacing x by $a+h$ in the expression for $f(x)$:

$$\begin{aligned} f(a+h) &= 2(a+h)^2 - 5(a+h) + 1 \\ &= 2(a^2 + 2ah + h^2) - 5(a+h) + 1 \\ &= 2a^2 + 4ah + 2h^2 - 5a - 5h + 1 \end{aligned}$$

Then we substitute into the given expression and simplify:

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(2a^2 + 4ah + 2h^2 - 5a - 5h + 1) - (2a^2 - 5a + 1)}{h} \\ &= \frac{2a^2 + 4ah + 2h^2 - 5a - 5h + 1 - 2a^2 + 5a - 1}{h} \\ &= \frac{4ah + 2h^2 - 5h}{h} = 4a + 2h - 5 \end{aligned}$$

The expression

$$\frac{f(a+h) - f(a)}{h}$$

in Example 3 is called a **difference quotient** and occurs frequently in calculus. As we will see in Chapter 2, it represents the average rate of change of $f(x)$ between $x = a$ and $x = a + h$.

Representations of Functions

There are four possible ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

If a single function can be represented in all four ways, it's often useful to go from one representation to another to gain additional insight into the function. (In Example 2, for instance, we started with algebraic formulas and then obtained the graphs.) But certain functions are described more naturally by one method than by another. With this in mind, let's reexamine the four situations that we considered at the beginning of this section.

- A. The most useful representation of the area of a circle as a function of its radius is probably the algebraic formula $A(r) = \pi r^2$, though it is possible to compile a table of values or to sketch a graph (half a parabola). Because a circle has to have a positive radius, the domain is $\{r \mid r > 0\} = (0, \infty)$, and the range is also $(0, \infty)$.
- B. We are given a description of the function in words: $P(t)$ is the human population of the world at time t . Let's measure t so that $t = 0$ corresponds to the year 1900. The table of values of world population provides a convenient representation of this function. If we plot these values, we get the graph (called a *scatter plot*) in Figure 9. It too is a useful representation; the graph allows us to absorb all the data at once. What about a formula? Of course, it's impossible to devise an explicit formula that gives the exact human population $P(t)$ at any time t . But it is possible to find an expression for a function that *approximates* $P(t)$. In fact, using methods explained in Section 1.2, we obtain the approximation

$$P(t) \approx f(t) = (1.43653 \times 10^9) \cdot (1.01395)^t$$

Figure 10 shows that it is a reasonably good “fit.” The function f is called a *mathematical model* for population growth. In other words, it is a function with an explicit formula that approximates the behavior of our given function. We will see, however, that the ideas of calculus can be applied to a table of values; an explicit formula is not necessary.

t	Population (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870

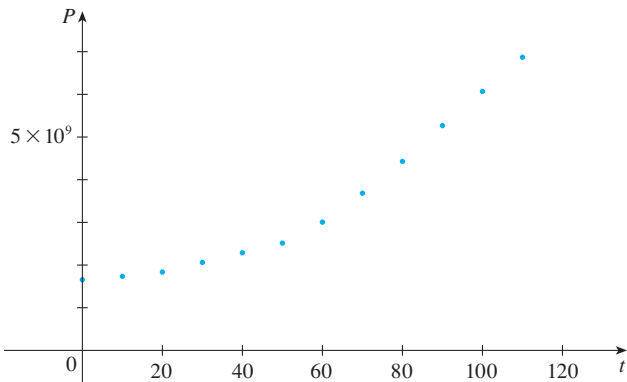


FIGURE 9

A function defined by a table of values is called a *tabular function*.

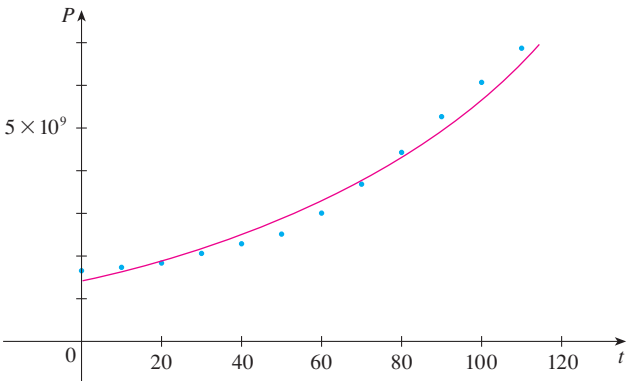


FIGURE 10

The function P is typical of the functions that arise whenever we attempt to apply calculus to the real world. We start with a verbal description of a function. Then we may be able to construct a table of values of the function, perhaps from instrument readings in a scientific experiment. Even though we don't have complete knowledge of the values of the function, we will see throughout the book that it is still possible to perform the operations of calculus on such a function.

w (ounces)	$C(w)$ (dollars)
$0 < w \leq 1$	0.88
$1 < w \leq 2$	1.05
$2 < w \leq 3$	1.22
$3 < w \leq 4$	1.39
$4 < w \leq 5$	1.56
\vdots	\vdots

- C. Again the function is described in words: Let $C(w)$ be the cost of mailing a large envelope with weight w . The rule that the US Postal Service used as of 2010 is as follows: The cost is 88 cents for up to 1 oz, plus 17 cents for each additional ounce (or less) up to 13 oz. The table of values shown in the margin is the most convenient representation for this function, though it is possible to sketch a graph (see Example 10).
- D. The graph shown in Figure 1 is the most natural representation of the vertical acceleration function $a(t)$. It's true that a table of values could be compiled, and it is even possible to devise an approximate formula. But everything a geologist needs to know—amplitudes and patterns—can be seen easily from the graph. (The same is true for the patterns seen in electrocardiograms of heart patients and polygraphs for lie-detection.)

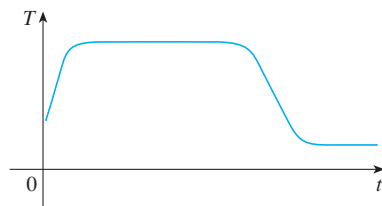


FIGURE 11

In the next example we sketch the graph of a function that is defined verbally.

EXAMPLE 4 When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running. Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

SOLUTION The initial temperature of the running water is close to room temperature because the water has been sitting in the pipes. When the water from the hot-water tank starts flowing from the faucet, T increases quickly. In the next phase, T is constant at the temperature of the heated water in the tank. When the tank is drained, T decreases to the temperature of the water supply. This enables us to make the rough sketch of T as a function of t in Figure 11.

In the following example we start with a verbal description of a function in a physical situation and obtain an explicit algebraic formula. The ability to do this is a useful skill in solving calculus problems that ask for the maximum or minimum values of quantities.

EXAMPLE 5 A rectangular storage container with an open top has a volume of 10 m^3 . The length of its base is twice its width. Material for the base costs \$10 per square meter; material for the sides costs \$6 per square meter. Express the cost of materials as a function of the width of the base.

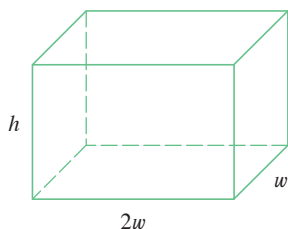


FIGURE 12

SOLUTION We draw a diagram as in Figure 12 and introduce notation by letting w and $2w$ be the width and length of the base, respectively, and h be the height.

The area of the base is $(2w)w = 2w^2$, so the cost, in dollars, of the material for the base is $10(2w^2)$. Two of the sides have area wh and the other two have area $2wh$, so the cost of the material for the sides is $6[2(wh) + 2(2wh)]$. The total cost is therefore

$$C = 10(2w^2) + 6[2(wh) + 2(2wh)] = 20w^2 + 36wh$$

To express C as a function of w alone, we need to eliminate h and we do so by using the fact that the volume is 10 m^3 . Thus

$$w(2w)h = 10$$

which gives

$$h = \frac{10}{2w^2} = \frac{5}{w^2}$$

Substituting this into the expression for C , we have

$$C = 20w^2 + 36w\left(\frac{5}{w^2}\right) = 20w^2 + \frac{180}{w}$$

Therefore the equation

$$C(w) = 20w^2 + \frac{180}{w} \quad w > 0$$

expresses C as a function of w .

EXAMPLE 6 Find the domain of each function.

(a) $f(x) = \sqrt{x+2}$

(b) $g(x) = \frac{1}{x^2 - x}$

SOLUTION

(a) Because the square root of a negative number is not defined (as a real number), the domain of f consists of all values of x such that $x + 2 \geq 0$. This is equivalent to $x \geq -2$, so the domain is the interval $[-2, \infty)$.

PS In setting up applied functions as in Example 5, it may be useful to review the principles of problem solving as discussed on page 75, particularly *Step 1: Understand the Problem*.

Domain Convention

If a function is given by a formula and the domain is not stated explicitly, the convention is that the domain is the set of all numbers for which the formula makes sense and defines a real number.

(b) Since

$$g(x) = \frac{1}{x^2 - x} = \frac{1}{x(x - 1)}$$

and division by 0 is not allowed, we see that $g(x)$ is not defined when $x = 0$ or $x = 1$. Thus the domain of g is

$$\{x \mid x \neq 0, x \neq 1\}$$

which could also be written in interval notation as

$$(-\infty, 0) \cup (0, 1) \cup (1, \infty)$$

The graph of a function is a curve in the xy -plane. But the question arises: Which curves in the xy -plane are graphs of functions? This is answered by the following test.

The Vertical Line Test A curve in the xy -plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

The reason for the truth of the Vertical Line Test can be seen in Figure 13. If each vertical line $x = a$ intersects a curve only once, at (a, b) , then exactly one functional value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and (a, c) , then the curve can't represent a function because a function can't assign two different values to a .

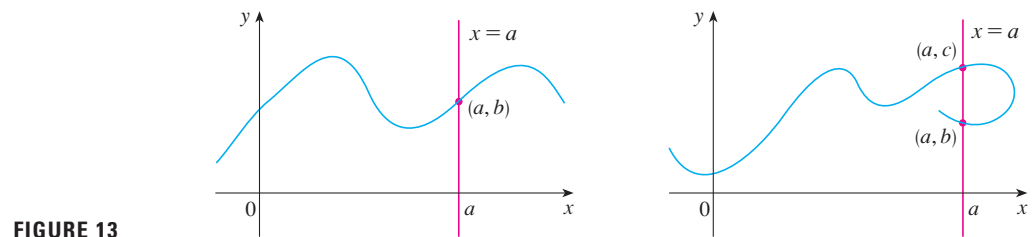


FIGURE 13

For example, the parabola $x = y^2 - 2$ shown in Figure 14(a) is not the graph of a function of x because, as you can see, there are vertical lines that intersect the parabola twice. The parabola, however, does contain the graphs of *two* functions of x . Notice that the equation $x = y^2 - 2$ implies $y^2 = x + 2$, so $y = \pm\sqrt{x + 2}$. Thus the upper and lower halves of the parabola are the graphs of the functions $f(x) = \sqrt{x + 2}$ [from Example 6(a)] and $g(x) = -\sqrt{x + 2}$. [See Figures 14(b) and (c).] We observe that if we reverse the roles of x and y , then the equation $x = h(y) = y^2 - 2$ *does* define x as a function of y (with y as the independent variable and x as the dependent variable) and the parabola now appears as the graph of the function h .

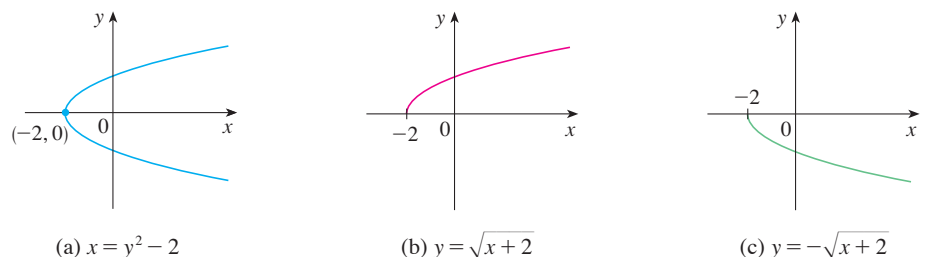


FIGURE 14

Piecewise Defined Functions

The functions in the following four examples are defined by different formulas in different parts of their domains. Such functions are called **piecewise defined functions**.

V EXAMPLE 7 A function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

SOLUTION Remember that a function is a rule. For this particular function the rule is the following: First look at the value of the input x . If it happens that $x \leq -1$, then the value of $f(x)$ is $1 - x$. On the other hand, if $x > -1$, then the value of $f(x)$ is x^2 .

$$\text{Since } -2 \leq -1, \text{ we have } f(-2) = 1 - (-2) = 3.$$

$$\text{Since } -1 \leq -1, \text{ we have } f(-1) = 1 - (-1) = 2.$$

$$\text{Since } 0 > -1, \text{ we have } f(0) = 0^2 = 0.$$

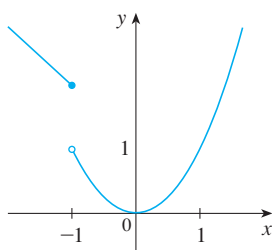


FIGURE 15

How do we draw the graph of f ? We observe that if $x \leq -1$, then $f(x) = 1 - x$, so the part of the graph of f that lies to the left of the vertical line $x = -1$ must coincide with the line $y = 1 - x$, which has slope -1 and y -intercept 1 . If $x > -1$, then $f(x) = x^2$, so the part of the graph of f that lies to the right of the line $x = -1$ must coincide with the graph of $y = x^2$, which is a parabola. This enables us to sketch the graph in Figure 15. The solid dot indicates that the point $(-1, 2)$ is included on the graph; the open dot indicates that the point $(-1, 1)$ is excluded from the graph.

The next example of a piecewise defined function is the absolute value function. Recall that the **absolute value** of a number a , denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0 , so we have

$$|a| \geq 0 \quad \text{for every number } a$$

For example,

$$|3| = 3 \quad |-3| = 3 \quad |0| = 0 \quad |\sqrt{2} - 1| = \sqrt{2} - 1 \quad |3 - \pi| = \pi - 3$$

In general, we have

$$\begin{aligned} |a| &= a & \text{if } a \geq 0 \\ |a| &= -a & \text{if } a < 0 \end{aligned}$$

(Remember that if a is negative, then $-a$ is positive.)

EXAMPLE 8 Sketch the graph of the absolute value function $f(x) = |x|$.

SOLUTION From the preceding discussion we know that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Using the same method as in Example 7, we see that the graph of f coincides with the line $y = x$ to the right of the y -axis and coincides with the line $y = -x$ to the left of the y -axis (see Figure 16).

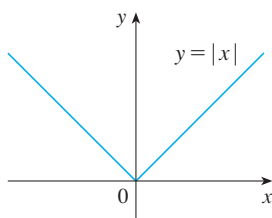


FIGURE 16

For a more extensive review of absolute values, see Appendix A.

EXAMPLE 9 Find a formula for the function f graphed in Figure 17.

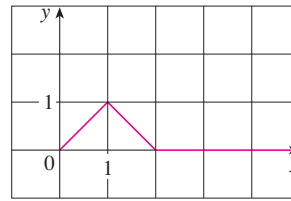


FIGURE 17

SOLUTION The line through $(0, 0)$ and $(1, 1)$ has slope $m = 1$ and y -intercept $b = 0$, so its equation is $y = x$. Thus, for the part of the graph of f that joins $(0, 0)$ to $(1, 1)$, we have

$$f(x) = x \quad \text{if } 0 \leq x \leq 1$$

Point-slope form of the equation of a line:

$$y - y_1 = m(x - x_1)$$

See Appendix B.

The line through $(1, 1)$ and $(2, 0)$ has slope $m = -1$, so its point-slope form is

$$y - 0 = (-1)(x - 2) \quad \text{or} \quad y = 2 - x$$

So we have

$$f(x) = 2 - x \quad \text{if } 1 < x \leq 2$$

We also see that the graph of f coincides with the x -axis for $x > 2$. Putting this information together, we have the following three-piece formula for f :

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } 1 < x \leq 2 \\ 0 & \text{if } x > 2 \end{cases}$$

EXAMPLE 10 In Example C at the beginning of this section we considered the cost $C(w)$ of mailing a large envelope with weight w . In effect, this is a piecewise defined function because, from the table of values on page 13, we have

$$C(w) = \begin{cases} 0.88 & \text{if } 0 < w \leq 1 \\ 1.05 & \text{if } 1 < w \leq 2 \\ 1.22 & \text{if } 2 < w \leq 3 \\ 1.39 & \text{if } 3 < w \leq 4 \\ \vdots & \end{cases}$$

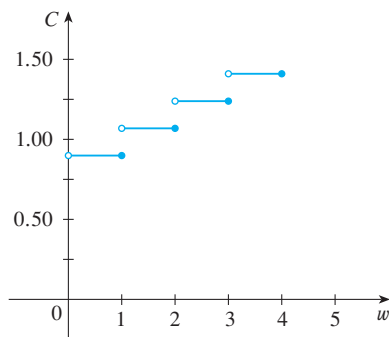


FIGURE 18

The graph is shown in Figure 18. You can see why functions similar to this one are called **step functions**—they jump from one value to the next. Such functions will be studied in Chapter 2.

Symmetry

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

The geometric significance of an even function is that its graph is symmetric with respect

to the y -axis (see Figure 19). This means that if we have plotted the graph of f for $x \geq 0$, we obtain the entire graph simply by reflecting this portion about the y -axis.

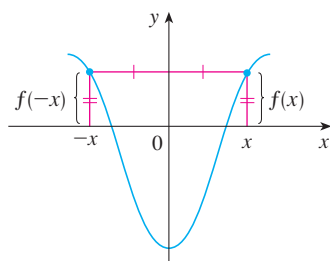


FIGURE 19 An even function

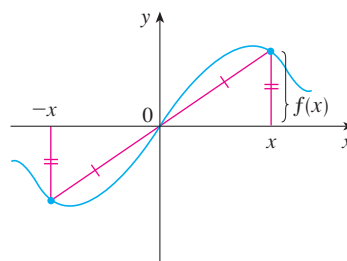


FIGURE 20 An odd function

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

The graph of an odd function is symmetric about the origin (see Figure 20). If we already have the graph of f for $x \geq 0$, we can obtain the entire graph by rotating this portion through 180° about the origin.

V EXAMPLE 11 Determine whether each of the following functions is even, odd, or neither even nor odd.

- (a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

SOLUTION

$$\begin{aligned} \text{(a)} \quad f(-x) &= (-x)^5 + (-x) = (-1)^5 x^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore f is an odd function.

$$\text{(b)} \quad g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

So g is even.

$$\text{(c)} \quad h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd.

The graphs of the functions in Example 11 are shown in Figure 21. Notice that the graph of h is symmetric neither about the y -axis nor about the origin.

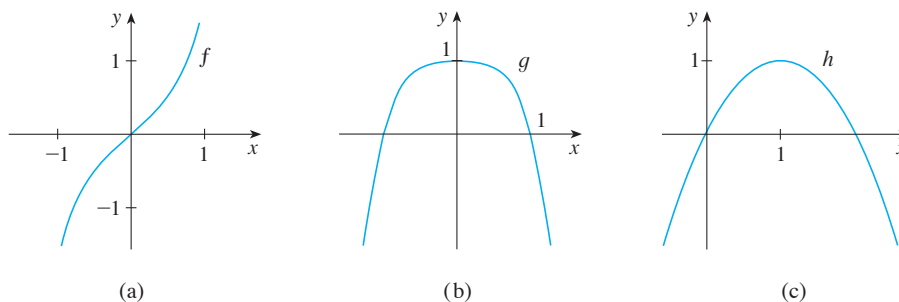


FIGURE 21

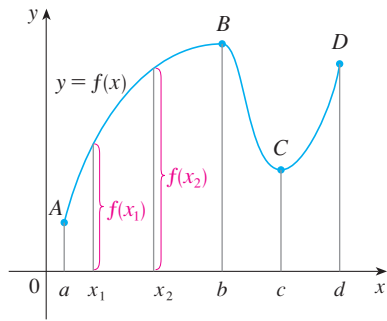


FIGURE 22

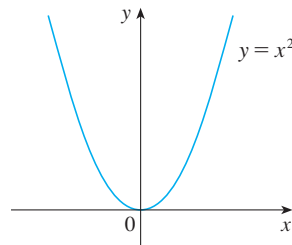


FIGURE 23

Increasing and Decreasing Functions

The graph shown in Figure 22 rises from A to B , falls from B to C , and rises again from C to D . The function f is said to be increasing on the interval $[a, b]$, decreasing on $[b, c]$, and increasing again on $[c, d]$. Notice that if x_1 and x_2 are any two numbers between a and b with $x_1 < x_2$, then $f(x_1) < f(x_2)$. We use this as the defining property of an increasing function.

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

In the definition of an increasing function it is important to realize that the inequality $f(x_1) < f(x_2)$ must be satisfied for *every* pair of numbers x_1 and x_2 in I with $x_1 < x_2$.

You can see from Figure 23 that the function $f(x) = x^2$ is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

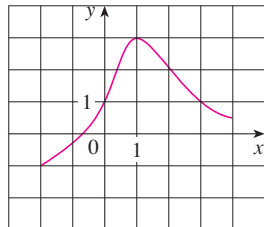
1.1 Exercises

- If $f(x) = x + \sqrt{2 - x}$ and $g(u) = u + \sqrt{2 - u}$, is it true that $f = g$?
- If

$$f(x) = \frac{x^2 - x}{x - 1} \quad \text{and} \quad g(x) = x$$

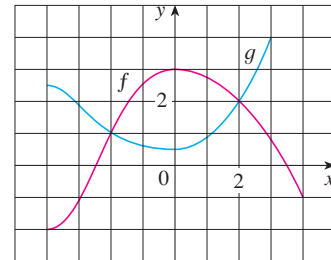
is it true that $f = g$?

- The graph of a function f is given.
 - State the value of $f(1)$.
 - Estimate the value of $f(-1)$.
 - For what values of x is $f(x) = 1$?
 - Estimate the value of x such that $f(x) = 0$.
 - State the domain and range of f .
 - On what interval is f increasing?



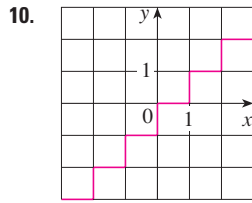
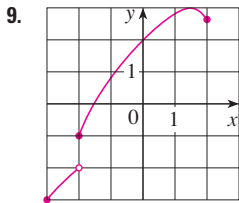
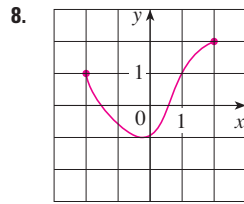
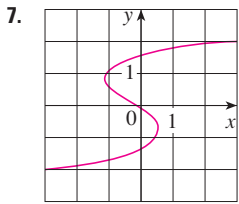
- The graphs of f and g are given.
 - State the values of $f(-4)$ and $g(3)$.
 - For what values of x is $f(x) = g(x)$?

- Estimate the solution of the equation $f(x) = -1$.
- On what interval is f decreasing?
- State the domain and range of f .
- State the domain and range of g .

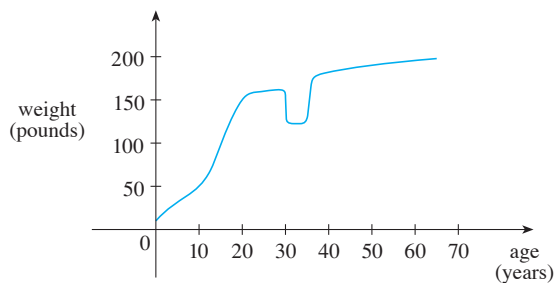


- Figure 1 was recorded by an instrument operated by the California Department of Mines and Geology at the University Hospital of the University of Southern California in Los Angeles. Use it to estimate the range of the vertical ground acceleration function at USC during the Northridge earthquake.
- In this section we discussed examples of ordinary, everyday functions: Population is a function of time, postage cost is a function of weight, water temperature is a function of time. Give three other examples of functions from everyday life that are described verbally. What can you say about the domain and range of each of your functions? If possible, sketch a rough graph of each function.

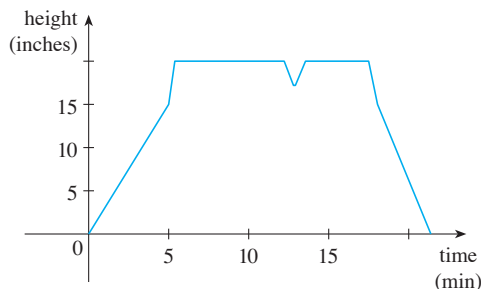
7–10 Determine whether the curve is the graph of a function of x . If it is, state the domain and range of the function.



11. The graph shown gives the weight of a certain person as a function of age. Describe in words how this person's weight varies over time. What do you think happened when this person was 30 years old?

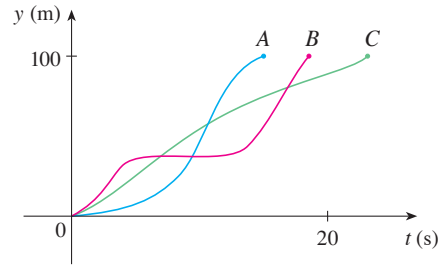


12. The graph shows the height of the water in a bathtub as a function of time. Give a verbal description of what you think happened.

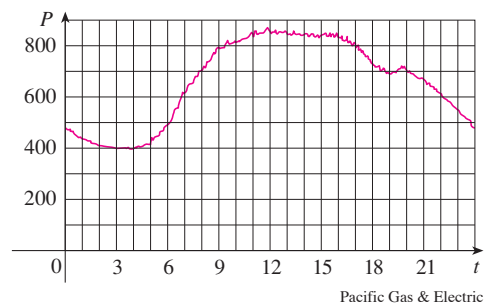


13. You put some ice cubes in a glass, fill the glass with cold water, and then let the glass sit on a table. Describe how the temperature of the water changes as time passes. Then sketch a rough graph of the temperature of the water as a function of the elapsed time.
14. Three runners compete in a 100-meter race. The graph depicts the distance run as a function of time for each runner. Describe

in words what the graph tells you about this race. Who won the race? Did each runner finish the race?



15. The graph shows the power consumption for a day in September in San Francisco. (P is measured in megawatts; t is measured in hours starting at midnight.)
- (a) What was the power consumption at 6 AM? At 6 PM?
- (b) When was the power consumption the lowest? When was it the highest? Do these times seem reasonable?



16. Sketch a rough graph of the number of hours of daylight as a function of the time of year.
17. Sketch a rough graph of the outdoor temperature as a function of time during a typical spring day.
18. Sketch a rough graph of the market value of a new car as a function of time for a period of 20 years. Assume the car is well maintained.
19. Sketch the graph of the amount of a particular brand of coffee sold by a store as a function of the price of the coffee.
20. You place a frozen pie in an oven and bake it for an hour. Then you take it out and let it cool before eating it. Describe how the temperature of the pie changes as time passes. Then sketch a rough graph of the temperature of the pie as a function of time.
21. A homeowner mows the lawn every Wednesday afternoon. Sketch a rough graph of the height of the grass as a function of time over the course of a four-week period.
22. An airplane takes off from an airport and lands an hour later at another airport, 400 miles away. If t represents the time in minutes since the plane has left the terminal building, let $x(t)$ be

the horizontal distance traveled and $y(t)$ be the altitude of the plane.

- Sketch a possible graph of $x(t)$.
- Sketch a possible graph of $y(t)$.
- Sketch a possible graph of the ground speed.
- Sketch a possible graph of the vertical velocity.

23. The number N (in millions) of US cellular phone subscribers is shown in the table. (Midyear estimates are given.)

t	1996	1998	2000	2002	2004	2006
N	44	69	109	141	182	233

- Use the data to sketch a rough graph of N as a function of t .
- Use your graph to estimate the number of cell-phone subscribers at midyear in 2001 and 2005.

24. Temperature readings T (in °F) were recorded every two hours from midnight to 2:00 PM in Phoenix on September 10, 2008. The time t was measured in hours from midnight.

t	0	2	4	6	8	10	12	14
T	82	75	74	75	84	90	93	94

- Use the readings to sketch a rough graph of T as a function of t .
- Use your graph to estimate the temperature at 9:00 AM.

25. If $f(x) = 3x^2 - x + 2$, find $f(2)$, $f(-2)$, $f(a)$, $f(-a)$, $f(a + 1)$, $2f(a)$, $f(a^2)$, $[f(a)]^2$, and $f(a + h)$.
26. A spherical balloon with radius r inches has volume $V(r) = \frac{4}{3}\pi r^3$. Find a function that represents the amount of air required to inflate the balloon from a radius of r inches to a radius of $r + 1$ inches.

- 27–30 Evaluate the difference quotient for the given function. Simplify your answer.

27. $f(x) = 4 + 3x - x^2$, $\frac{f(3 + h) - f(3)}{h}$

28. $f(x) = x^3$, $\frac{f(a + h) - f(a)}{h}$

29. $f(x) = \frac{1}{x}$, $\frac{f(x) - f(a)}{x - a}$

30. $f(x) = \frac{x + 3}{x + 1}$, $\frac{f(x) - f(1)}{x - 1}$

- 31–37 Find the domain of the function.

31. $f(x) = \frac{x + 4}{x^2 - 9}$

32. $f(x) = \frac{2x^3 - 5}{x^2 + x - 6}$

33. $f(t) = \sqrt[3]{2t - 1}$

34. $g(t) = \sqrt{3 - t} - \sqrt{2 + t}$

35. $h(x) = \frac{1}{\sqrt[4]{x^2 - 5x}}$

36. $f(u) = \frac{u + 1}{1 + \frac{1}{u + 1}}$

37. $F(p) = \sqrt{2 - \sqrt{p}}$

38. Find the domain and range and sketch the graph of the function $h(x) = \sqrt{4 - x^2}$.

- 39–50 Find the domain and sketch the graph of the function.

39. $f(x) = 2 - 0.4x$

40. $F(x) = x^2 - 2x + 1$

41. $f(t) = 2t + t^2$

42. $H(t) = \frac{4 - t^2}{2 - t}$

43. $g(x) = \sqrt{x - 5}$

44. $F(x) = |2x + 1|$

45. $G(x) = \frac{3x + |x|}{x}$

46. $g(x) = |x| - x$

47. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 1 - x & \text{if } x \geq 0 \end{cases}$

48. $f(x) = \begin{cases} 3 - \frac{1}{2}x & \text{if } x \leq 2 \\ 2x - 5 & \text{if } x > 2 \end{cases}$

49. $f(x) = \begin{cases} x + 2 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$

50. $f(x) = \begin{cases} x + 9 & \text{if } x < -3 \\ -2x & \text{if } |x| \leq 3 \\ -6 & \text{if } x > 3 \end{cases}$

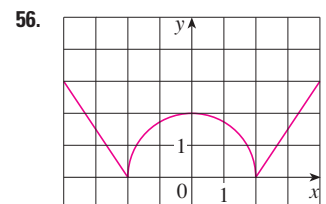
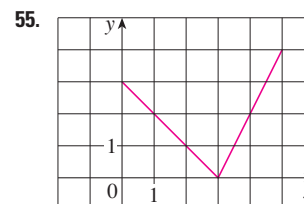
- 51–56 Find an expression for the function whose graph is the given curve.

51. The line segment joining the points $(1, -3)$ and $(5, 7)$

52. The line segment joining the points $(-5, 10)$ and $(7, -10)$

53. The bottom half of the parabola $x + (y - 1)^2 = 0$

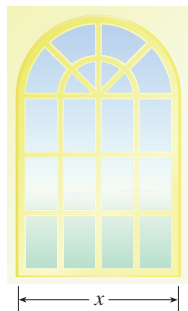
54. The top half of the circle $x^2 + (y - 2)^2 = 4$



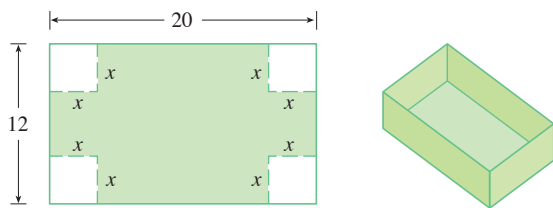
- 57–61 Find a formula for the described function and state its domain.

57. A rectangle has perimeter 20 m. Express the area of the rectangle as a function of the length of one of its sides.

58. A rectangle has area 16 m^2 . Express the perimeter of the rectangle as a function of the length of one of its sides.
59. Express the area of an equilateral triangle as a function of the length of a side.
60. Express the surface area of a cube as a function of its volume.
61. An open rectangular box with volume 2 m^3 has a square base. Express the surface area of the box as a function of the length of a side of the base.
62. A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 ft, express the area A of the window as a function of the width x of the window.



63. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x .

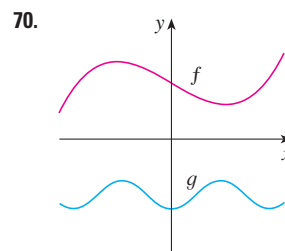
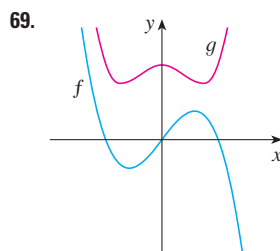


64. A cell phone plan has a basic charge of \$35 a month. The plan includes 400 free minutes and charges 10 cents for each additional minute of usage. Write the monthly cost C as a function of the number x of minutes used and graph C as a function of x for $0 \leq x \leq 600$.
65. In a certain state the maximum speed permitted on freeways is 65 mi/h and the minimum speed is 40 mi/h. The fine for violating these limits is \$15 for every mile per hour above the maximum speed or below the minimum speed. Express the amount of the fine F as a function of the driving speed x and graph $F(x)$ for $0 \leq x \leq 100$.
66. An electricity company charges its customers a base rate of \$10 a month, plus 6 cents per kilowatt-hour (kWh) for the first 1200 kWh and 7 cents per kWh for all usage over 1200 kWh. Express the monthly cost E as a function of the amount x of electricity used. Then graph the function E for $0 \leq x \leq 2000$.

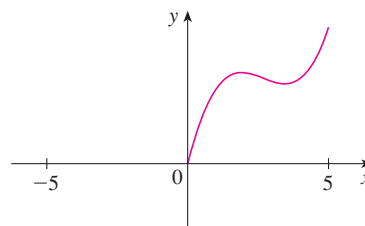
67. In a certain country, income tax is assessed as follows. There is no tax on income up to \$10,000. Any income over \$10,000 is taxed at a rate of 10%, up to an income of \$20,000. Any income over \$20,000 is taxed at 15%.
- (a) Sketch the graph of the tax rate R as a function of the income I .
- (b) How much tax is assessed on an income of \$14,000? On \$26,000?
- (c) Sketch the graph of the total assessed tax T as a function of the income I .

68. The functions in Example 10 and Exercise 67 are called *step functions* because their graphs look like stairs. Give two other examples of step functions that arise in everyday life.

69–70 Graphs of f and g are shown. Decide whether each function is even, odd, or neither. Explain your reasoning.



71. (a) If the point $(5, 3)$ is on the graph of an even function, what other point must also be on the graph?
- (b) If the point $(5, 3)$ is on the graph of an odd function, what other point must also be on the graph?
72. A function f has domain $[-5, 5]$ and a portion of its graph is shown.
- (a) Complete the graph of f if it is known that f is even.
- (b) Complete the graph of f if it is known that f is odd.



73–78 Determine whether f is even, odd, or neither. If you have a graphing calculator, use it to check your answer visually.

73. $f(x) = \frac{x}{x^2 + 1}$

74. $f(x) = \frac{x^2}{x^4 + 1}$

75. $f(x) = \frac{x}{x + 1}$

76. $f(x) = x|x|$

77. $f(x) = 1 + 3x^2 - x^4$

78. $f(x) = 1 + 3x^3 - x^5$

79. If f and g are both even functions, is $f + g$ even? If f and g are both odd functions, is $f + g$ odd? What if f is even and g is odd? Justify your answers.
80. If f and g are both even functions, is the product fg even? If f and g are both odd functions, is fg odd? What if f is even and g is odd? Justify your answers.

1.2 Mathematical Models: A Catalog of Essential Functions

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as the size of a population, the demand for a product, the speed of a falling object, the concentration of a product in a chemical reaction, the life expectancy of a person at birth, or the cost of emission reductions. The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

Figure 1 illustrates the process of mathematical modeling. Given a real-world problem, our first task is to formulate a mathematical model by identifying and naming the independent and dependent variables and making assumptions that simplify the phenomenon enough to make it mathematically tractable. We use our knowledge of the physical situation and our mathematical skills to obtain equations that relate the variables. In situations where there is no physical law to guide us, we may need to collect data (either from a library or the Internet or by conducting our own experiments) and examine the data in the form of a table in order to discern patterns. From this numerical representation of a function we may wish to obtain a graphical representation by plotting the data. The graph might even suggest a suitable algebraic formula in some cases.

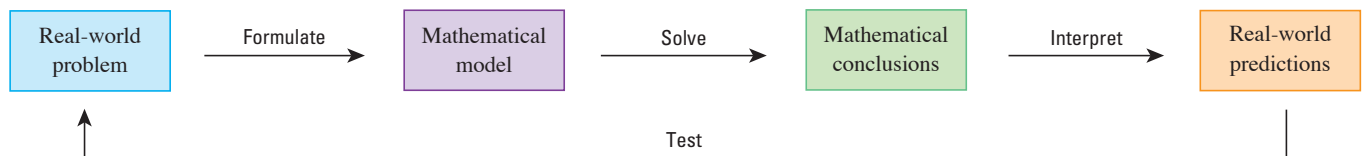


FIGURE 1 The modeling process

The second stage is to apply the mathematics that we know (such as the calculus that will be developed throughout this book) to the mathematical model that we have formulated in order to derive mathematical conclusions. Then, in the third stage, we take those mathematical conclusions and interpret them as information about the original real-world phenomenon by way of offering explanations or making predictions. The final step is to test our predictions by checking against new real data. If the predictions don't compare well with reality, we need to refine our model or to formulate a new model and start the cycle again.

A mathematical model is never a completely accurate representation of a physical situation—it is an *idealization*. A good model simplifies reality enough to permit mathematical calculations but is accurate enough to provide valuable conclusions. It is important to realize the limitations of the model. In the end, Mother Nature has the final say.

There are many different types of functions that can be used to model relationships observed in the real world. In what follows, we discuss the behavior and graphs of these functions and give examples of situations appropriately modeled by such functions.

Linear Models

When we say that y is a **linear function** of x , we mean that the graph of the function is a line, so we can use the slope-intercept form of the equation of a line to write a formula for

the function as

$$y = f(x) = mx + b$$

where m is the slope of the line and b is the y -intercept.

A characteristic feature of linear functions is that they grow at a constant rate. For instance, Figure 2 shows a graph of the linear function $f(x) = 3x - 2$ and a table of sample values. Notice that whenever x increases by 0.1, the value of $f(x)$ increases by 0.3. So $f(x)$ increases three times as fast as x . Thus the slope of the graph $y = 3x - 2$, namely 3, can be interpreted as the rate of change of y with respect to x .

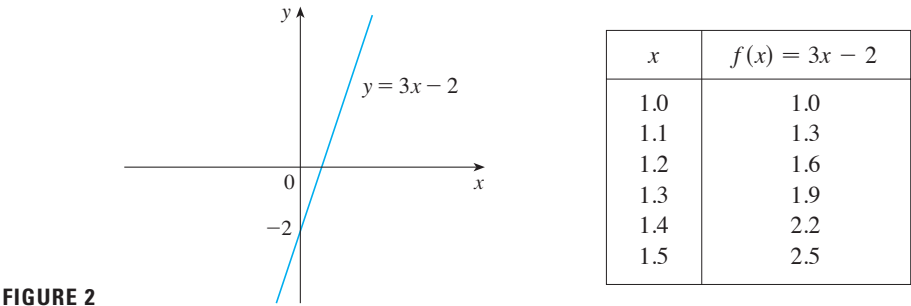


FIGURE 2

V EXAMPLE 1

- (a) As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C , express the temperature T (in $^{\circ}\text{C}$) as a function of the height h (in kilometers), assuming that a linear model is appropriate.
- (b) Draw the graph of the function in part (a). What does the slope represent?
- (c) What is the temperature at a height of 2.5 km?

SOLUTION

- (a) Because we are assuming that T is a linear function of h , we can write

$$T = mh + b$$

We are given that $T = 20$ when $h = 0$, so

$$20 = m \cdot 0 + b = b$$

In other words, the y -intercept is $b = 20$.

We are also given that $T = 10$ when $h = 1$, so

$$10 = m \cdot 1 + 20$$

The slope of the line is therefore $m = 10 - 20 = -10$ and the required linear function is

$$T = -10h + 20$$

- (b) The graph is sketched in Figure 3. The slope is $m = -10^{\circ}\text{C}/\text{km}$, and this represents the rate of change of temperature with respect to height.
- (c) At a height of $h = 2.5$ km, the temperature is

$$T = -10(2.5) + 20 = -5^{\circ}\text{C}$$

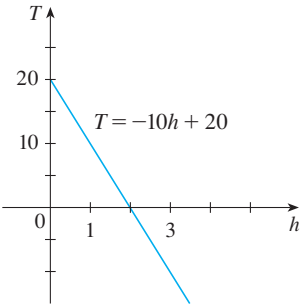


FIGURE 3

If there is no physical law or principle to help us formulate a model, we construct an **empirical model**, which is based entirely on collected data. We seek a curve that “fits” the data in the sense that it captures the basic trend of the data points.

V EXAMPLE 2 Table 1 lists the average carbon dioxide level in the atmosphere, measured in parts per million at Mauna Loa Observatory from 1980 to 2008. Use the data in Table 1 to find a model for the carbon dioxide level.

SOLUTION We use the data in Table 1 to make the scatter plot in Figure 4, where t represents time (in years) and C represents the CO₂ level (in parts per million, ppm).

TABLE 1			
Year	CO ₂ level (in ppm)	Year	CO ₂ level (in ppm)
1980	338.7	1996	362.4
1982	341.2	1998	366.5
1984	344.4	2000	369.4
1986	347.2	2002	373.2
1988	351.5	2004	377.5
1990	354.2	2006	381.9
1992	356.3	2008	385.6
1994	358.6		

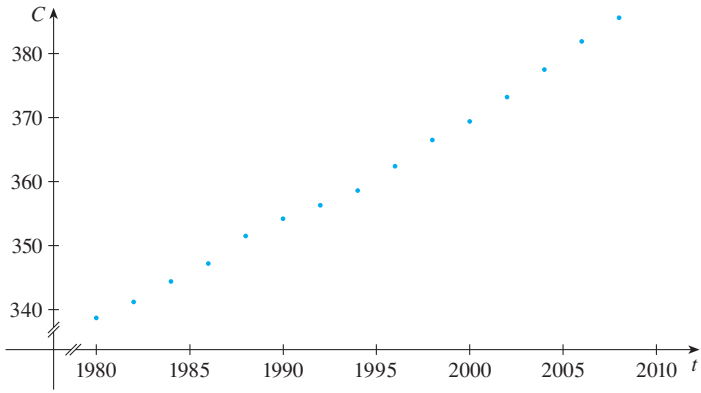


FIGURE 4 Scatter plot for the average CO₂ level

Notice that the data points appear to lie close to a straight line, so it’s natural to choose a linear model in this case. But there are many possible lines that approximate these data points, so which one should we use? One possibility is the line that passes through the first and last data points. The slope of this line is

$$\frac{385.6 - 338.7}{2008 - 1980} = \frac{46.9}{28} = 1.675$$

and its equation is

$$C - 338.7 = 1.675(t - 1980)$$

or

1

$$C = 1.675t - 2977.8$$

Equation 1 gives one possible linear model for the carbon dioxide level; it is graphed in Figure 5.

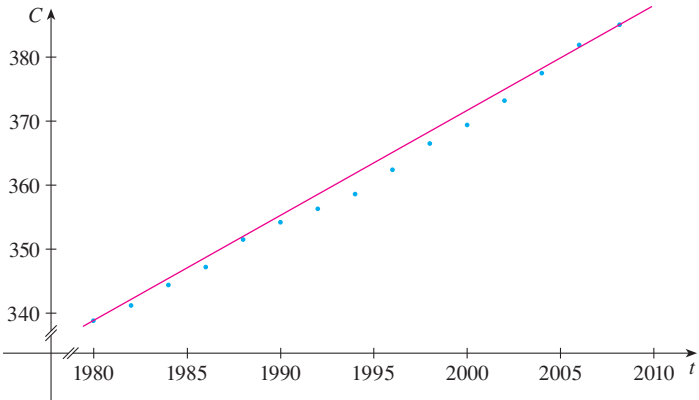


FIGURE 5
Linear model through
first and last data points

A computer or graphing calculator finds the regression line by the method of **least squares**, which is to minimize the sum of the squares of the vertical distances between the data points and the line. The details are explained in Section 14.7.

Notice that our model gives values higher than most of the actual CO₂ levels. A better linear model is obtained by a procedure from statistics called *linear regression*. If we use a graphing calculator, we enter the data from Table 1 into the data editor and choose the linear regression command. (With Maple we use the `fit[leastsquare]` command in the stats package; with Mathematica we use the `Fit` command.) The machine gives the slope and y-intercept of the regression line as

$$m = 1.65429 \quad b = -2938.07$$

So our least squares model for the CO₂ level is

$$\boxed{2} \quad C = 1.65429t - 2938.07$$

In Figure 6 we graph the regression line as well as the data points. Comparing with Figure 5, we see that it gives a better fit than our previous linear model.

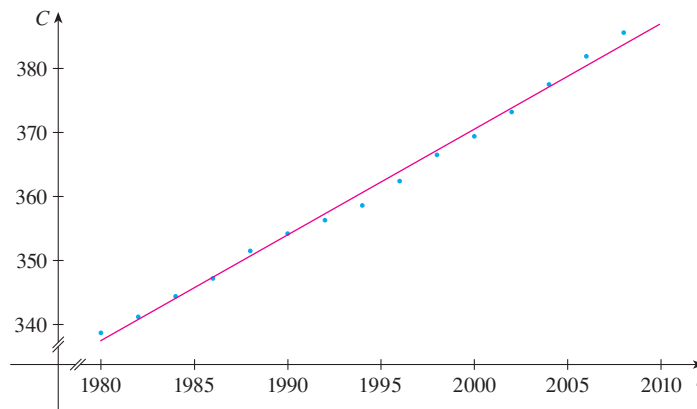


FIGURE 6
The regression line

V EXAMPLE 3 Use the linear model given by Equation 2 to estimate the average CO₂ level for 1987 and to predict the level for the year 2015. According to this model, when will the CO₂ level exceed 420 parts per million?

SOLUTION Using Equation 2 with $t = 1987$, we estimate that the average CO₂ level in 1987 was

$$C(1987) = (1.65429)(1987) - 2938.07 \approx 349.00$$

This is an example of *interpolation* because we have estimated a value *between* observed values. (In fact, the Mauna Loa Observatory reported that the average CO₂ level in 1987 was 348.93 ppm, so our estimate is quite accurate.)

With $t = 2015$, we get

$$C(2015) = (1.65429)(2015) - 2938.07 \approx 395.32$$

So we predict that the average CO₂ level in the year 2015 will be 395.3 ppm. This is an example of *extrapolation* because we have predicted a value *outside* the region of observations. Consequently, we are far less certain about the accuracy of our prediction.

Using Equation 2, we see that the CO₂ level exceeds 420 ppm when

$$1.65429t - 2938.07 > 420$$

Solving this inequality, we get

$$t > \frac{3358.07}{1.65429} \approx 2029.92$$

We therefore predict that the CO₂ level will exceed 420 ppm by the year 2030. This prediction is risky because it involves a time quite remote from our observations. In fact, we see from Figure 6 that the trend has been for CO₂ levels to increase rather more rapidly in recent years, so the level might exceed 420 ppm well before 2030.

Polynomials

A function P is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \dots, a_n$ are constants called the **coefficients** of the polynomial. The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the **degree** of the polynomial is n . For example, the function

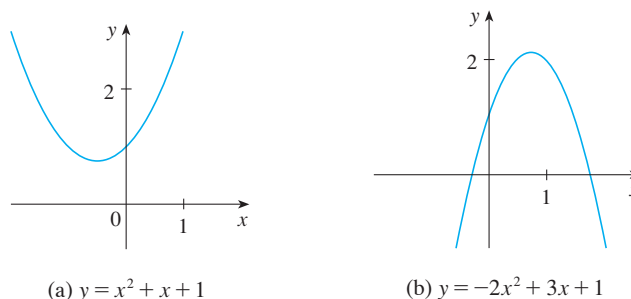
$$P(x) = 2x^6 - x^4 + \frac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a linear function. A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function**. Its graph is always a parabola obtained by shifting the parabola $y = ax^2$, as we will see in the next section. The parabola opens upward if $a > 0$ and downward if $a < 0$. (See Figure 7.)

FIGURE 7

The graphs of quadratic functions are parabolas.



A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d \quad a \neq 0$$

and is called a **cubic function**. Figure 8 shows the graph of a cubic function in part (a) and graphs of polynomials of degrees 4 and 5 in parts (b) and (c). We will see later why the graphs have these shapes.

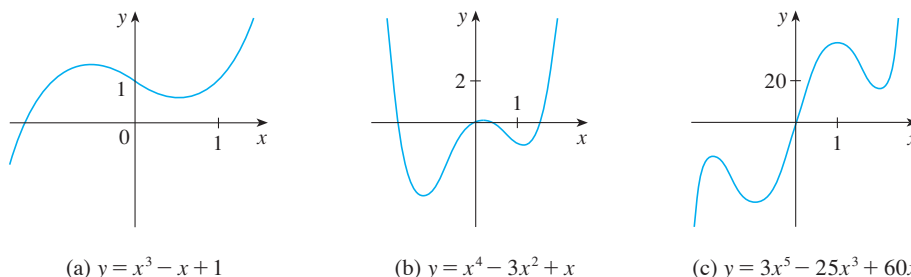


FIGURE 8

Polynomials are commonly used to model various quantities that occur in the natural and social sciences. For instance, in Section 3.7 we will explain why economists often use a polynomial $P(x)$ to represent the cost of producing x units of a commodity. In the following example we use a quadratic function to model the fall of a ball.

TABLE 2

Time (seconds)	Height (meters)
0	450
1	445
2	431
3	408
4	375
5	332
6	279
7	216
8	143
9	61

EXAMPLE 4 A ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground, and its height h above the ground is recorded at 1-second intervals in Table 2. Find a model to fit the data and use the model to predict the time at which the ball hits the ground.

SOLUTION We draw a scatter plot of the data in Figure 9 and observe that a linear model is inappropriate. But it looks as if the data points might lie on a parabola, so we try a quadratic model instead. Using a graphing calculator or computer algebra system (which uses the least squares method), we obtain the following quadratic model:

3

$$h = 449.36 + 0.96t - 4.90t^2$$

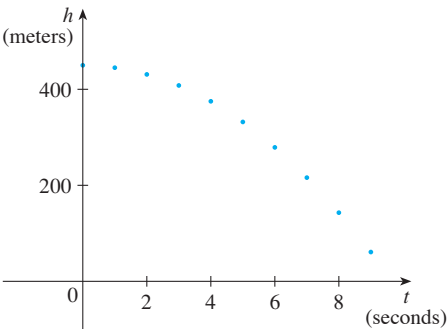


FIGURE 9
Scatter plot for a falling ball

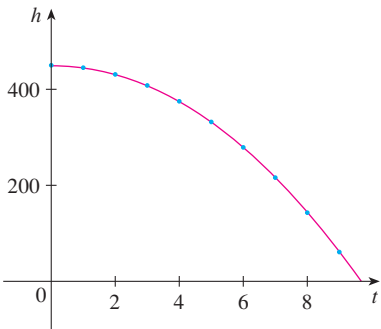


FIGURE 10
Quadratic model for a falling ball

In Figure 10 we plot the graph of Equation 3 together with the data points and see that the quadratic model gives a very good fit. The ball hits the ground when $h = 0$, so we solve the quadratic equation

$$-4.90t^2 + 0.96t + 449.36 = 0$$

The quadratic formula gives

$$t = \frac{-0.96 \pm \sqrt{(0.96)^2 - 4(-4.90)(449.36)}}{2(-4.90)}$$

The positive root is $t \approx 9.67$, so we predict that the ball will hit the ground after about 9.7 seconds.

Power Functions

A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**. We consider several cases.

(i) $a = n$, where n is a positive integer

The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4$, and 5 are shown in Figure 11. (These are polynomials with only one term.) We already know the shape of the graphs of $y = x$ (a line through the origin with slope 1) and $y = x^2$ [a parabola, see Example 2(b) in Section 1.1].

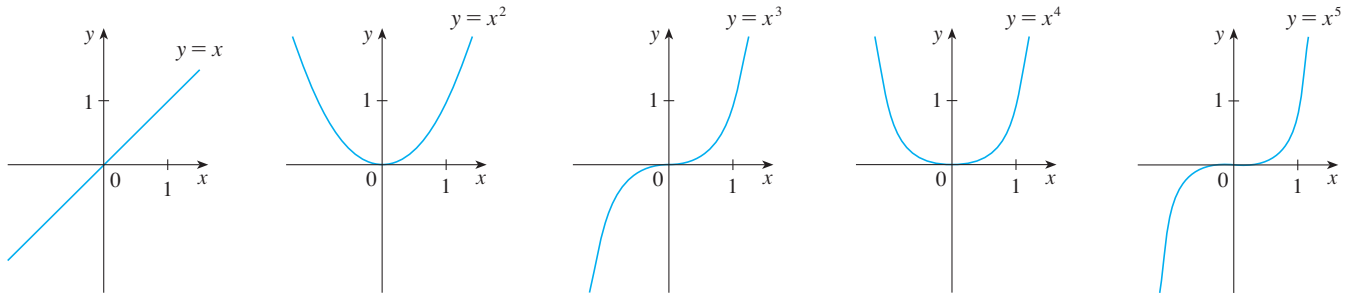


FIGURE 11 Graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4, 5$

The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd. If n is even, then $f(x) = x^n$ is an even function and its graph is similar to the parabola $y = x^2$. If n is odd, then $f(x) = x^n$ is an odd function and its graph is similar to that of $y = x^3$. Notice from Figure 12, however, that as n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| \geq 1$. (If x is small, then x^2 is smaller, x^3 is even smaller, x^4 is smaller still, and so on.)

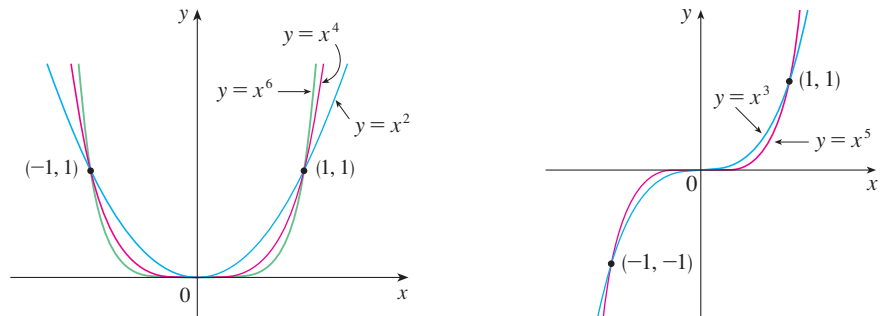


FIGURE 12
Families of power functions

(ii) $a = 1/n$, where n is a positive integer

The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**. For $n = 2$ it is the square root function $f(x) = \sqrt{x}$, whose domain is $[0, \infty)$ and whose graph is the upper half of the parabola $x = y^2$. [See Figure 13(a).] For other even values of n , the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$. For $n = 3$ we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is \mathbb{R} (recall that every real number has a cube root) and whose graph is shown in Figure 13(b). The graph of $y = \sqrt[n]{x}$ for n odd ($n > 3$) is similar to that of $y = \sqrt[3]{x}$.

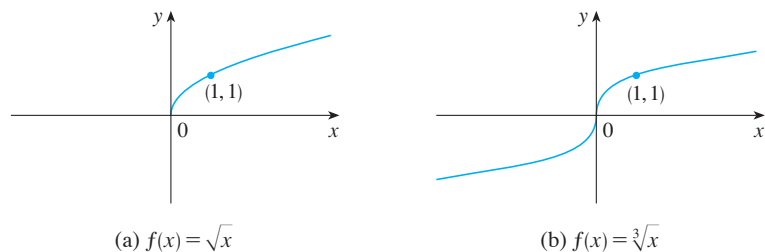


FIGURE 13
Graphs of root functions

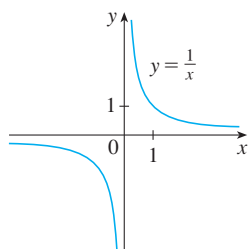


FIGURE 14

The reciprocal function

(iii) $a = -1$

The graph of the **reciprocal function** $f(x) = x^{-1} = 1/x$ is shown in Figure 14. Its graph has the equation $y = 1/x$, or $xy = 1$, and is a hyperbola with the coordinate axes as its asymptotes. This function arises in physics and chemistry in connection with Boyle's Law, which says that, when the temperature is constant, the volume V of a gas is inversely proportional to the pressure P :

$$V = \frac{C}{P}$$

where C is a constant. Thus the graph of V as a function of P (see Figure 15) has the same general shape as the right half of Figure 14.

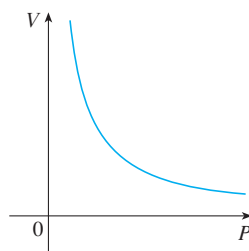


FIGURE 15

Volume as a function of pressure
at constant temperature

Power functions are also used to model species-area relationships (Exercises 26–27), illumination as a function of a distance from a light source (Exercise 25), and the period of revolution of a planet as a function of its distance from the sun (Exercise 28).

Rational Functions

A **rational function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain consists of all values of x such that $Q(x) \neq 0$. A simple example of a rational function is the function $f(x) = 1/x$, whose domain is $\{x \mid x \neq 0\}$; this is the reciprocal function graphed in Figure 14. The function

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

is a rational function with domain $\{x \mid x \neq \pm 2\}$. Its graph is shown in Figure 16.

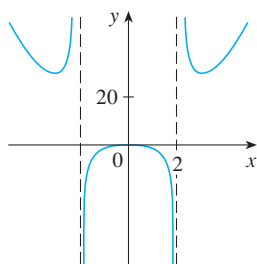


FIGURE 16

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

Algebraic Functions

A function f is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials. Any rational function is automatically an algebraic function. Here are two more examples:

$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

When we sketch algebraic functions in Chapter 4, we will see that their graphs can assume a variety of shapes. Figure 17 illustrates some of the possibilities.

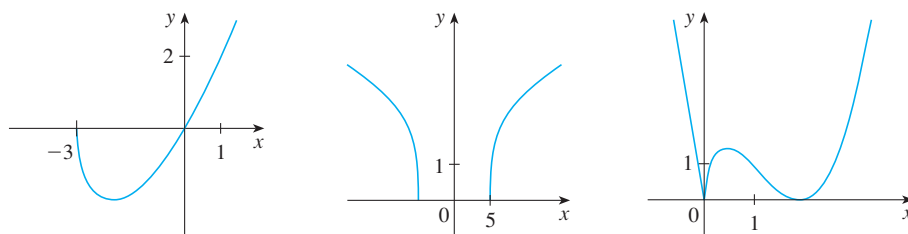


FIGURE 17

(a) $f(x) = x\sqrt{x+3}$

(b) $g(x) = \sqrt[4]{x^2 - 25}$

(c) $h(x) = x^{2/3}(x-2)^2$

An example of an algebraic function occurs in the theory of relativity. The mass of a particle with velocity v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and $c = 3.0 \times 10^8$ km/s is the speed of light in a vacuum.

Trigonometric Functions

The Reference Pages are located at the front and back of the book.

Trigonometry and the trigonometric functions are reviewed on Reference Page 2 and also in Appendix D. In calculus the convention is that radian measure is always used (except when otherwise indicated). For example, when we use the function $f(x) = \sin x$, it is understood that $\sin x$ means the sine of the angle whose radian measure is x . Thus the graphs of the sine and cosine functions are as shown in Figure 18.

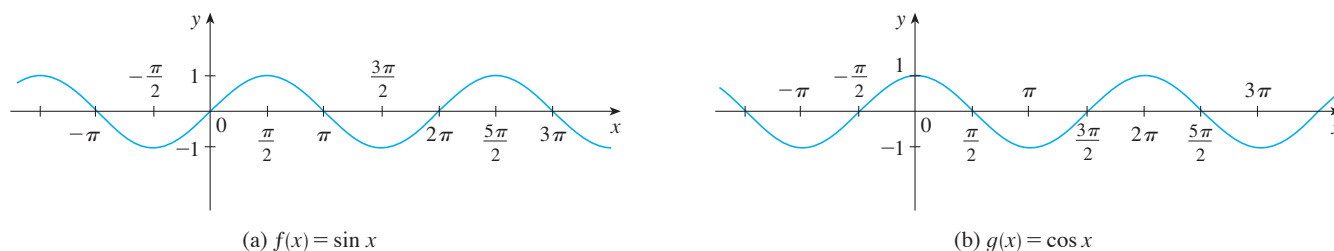


FIGURE 18

Notice that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus, for all values of x , we have

$$-1 \leq \sin x \leq 1 \quad -1 \leq \cos x \leq 1$$

or, in terms of absolute values,

$$|\sin x| \leq 1 \quad |\cos x| \leq 1$$

Also, the zeros of the sine function occur at the integer multiples of π ; that is,

$$\sin x = 0 \quad \text{when} \quad x = n\pi \quad n \text{ an integer}$$

An important property of the sine and cosine functions is that they are periodic functions and have period 2π . This means that, for all values of x ,

$$\sin(x + 2\pi) = \sin x \quad \cos(x + 2\pi) = \cos x$$

The periodic nature of these functions makes them suitable for modeling repetitive phenomena such as tides, vibrating springs, and sound waves. For instance, in Example 4 in Section 1.3 we will see that a reasonable model for the number of hours of daylight in Philadelphia t days after January 1 is given by the function

$$L(t) = 12 + 2.8 \sin \left[\frac{2\pi}{365}(t - 80) \right]$$

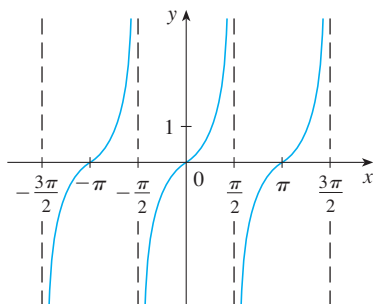


FIGURE 19

$y = \tan x$

The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

and its graph is shown in Figure 19. It is undefined whenever $\cos x = 0$, that is, when $x = \pm\pi/2, \pm3\pi/2, \dots$. Its range is $(-\infty, \infty)$. Notice that the tangent function has period π :

$$\tan(x + \pi) = \tan x \quad \text{for all } x$$

The remaining three trigonometric functions (cosecant, secant, and cotangent) are the reciprocals of the sine, cosine, and tangent functions. Their graphs are shown in Appendix D.

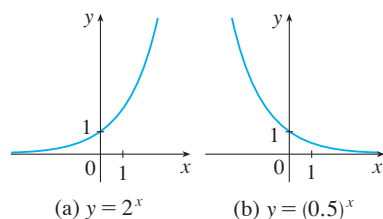


FIGURE 20

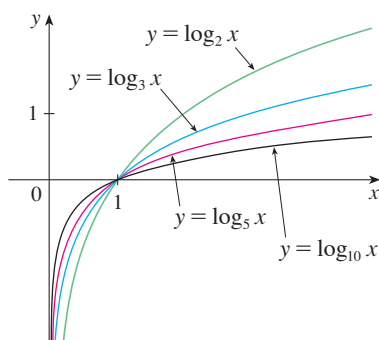


FIGURE 21

Exponential Functions

The **exponential functions** are the functions of the form $f(x) = a^x$, where the base a is a positive constant. The graphs of $y = 2^x$ and $y = (0.5)^x$ are shown in Figure 20. In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

Exponential functions will be studied in detail in Section 1.5, and we will see that they are useful for modeling many natural phenomena, such as population growth (if $a > 1$) and radioactive decay (if $a < 1$).

Logarithmic Functions

The **logarithmic functions** $f(x) = \log_a x$, where the base a is a positive constant, are the inverse functions of the exponential functions. They will be studied in Section 1.6. Figure 21 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when $x > 1$.

EXAMPLE 5 Classify the following functions as one of the types of functions that we have discussed.

(a) $f(x) = 5^x$

(b) $g(x) = x^5$

(c) $h(x) = \frac{1+x}{1-\sqrt{x}}$

(d) $u(t) = 1 - t + 5t^4$

SOLUTION

(a) $f(x) = 5^x$ is an exponential function. (The x is the exponent.)

(b) $g(x) = x^5$ is a power function. (The x is the base.) We could also consider it to be a polynomial of degree 5.

(c) $h(x) = \frac{1+x}{1-\sqrt{x}}$ is an algebraic function.

(d) $u(t) = 1 - t + 5t^4$ is a polynomial of degree 4.

1.2 Exercises

1–2 Classify each function as a power function, root function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function.

- (a) $f(x) = \log_2 x$ (b) $g(x) = \sqrt[4]{x}$

(c) $h(x) = \frac{2x^3}{1-x^2}$ (d) $u(t) = 1 - 1.1t + 2.54t^2$

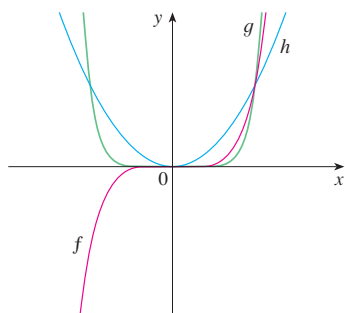
(e) $v(t) = 5^t$ (f) $w(\theta) = \sin \theta \cos^2 \theta$
- (a) $y = \pi^x$ (b) $y = x^\pi$

(c) $y = x^2(2 - x^3)$ (d) $y = \tan t - \cos t$

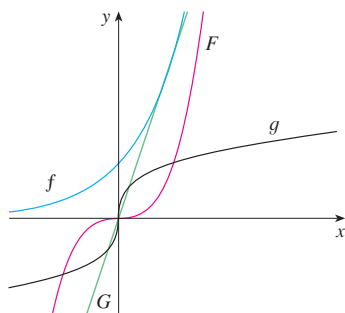
(e) $y = \frac{s}{1+s}$ (f) $y = \frac{\sqrt{x^3 - 1}}{1 + \sqrt[3]{x}}$

3–4 Match each equation with its graph. Explain your choices. (Don't use a computer or graphing calculator.)

- 3.** (a) $y = x^2$ (b) $y = x^5$ (c) $y = x^8$

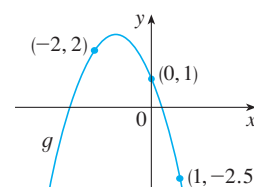
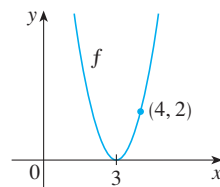


- 4.** (a) $y = 3x$ (b) $y = 3^x$
(c) $y = x^3$ (d) $y = \sqrt[3]{x}$



- 5.** (a) Find an equation for the family of linear functions with slope 2 and sketch several members of the family.
(b) Find an equation for the family of linear functions such that $f(2) = 1$ and sketch several members of the family.
(c) Which function belongs to both families?

- 6.** What do all members of the family of linear functions $f(x) = 1 + m(x + 3)$ have in common? Sketch several members of the family.
- 7.** What do all members of the family of linear functions $f(x) = c - x$ have in common? Sketch several members of the family.
- 8.** Find expressions for the quadratic functions whose graphs are shown.



- 9.** Find an expression for a cubic function f if $f(1) = 6$ and $f(-1) = f(0) = f(2) = 0$.
- 10.** Recent studies indicate that the average surface temperature of the earth has been rising steadily. Some scientists have modeled the temperature by the linear function $T = 0.02t + 8.50$, where T is temperature in $^{\circ}\text{C}$ and t represents years since 1900.
- What do the slope and T -intercept represent?
 - Use the equation to predict the average global surface temperature in 2100.
- 11.** If the recommended adult dosage for a drug is D (in mg), then to determine the appropriate dosage c for a child of age a , pharmacists use the equation $c = 0.0417D(a + 1)$. Suppose the dosage for an adult is 200 mg.
- Find the slope of the graph of c . What does it represent?
 - What is the dosage for a newborn?
- 12.** The manager of a weekend flea market knows from past experience that if he charges x dollars for a rental space at the market, then the number y of spaces he can rent is given by the equation $y = 200 - 4x$.
- Sketch a graph of this linear function. (Remember that the rental charge per space and the number of spaces rented can't be negative quantities.)
 - What do the slope, the y -intercept, and the x -intercept of the graph represent?
- 13.** The relationship between the Fahrenheit (F) and Celsius (C) temperature scales is given by the linear function $F = \frac{9}{5}C + 32$.
- Sketch a graph of this function.
 - What is the slope of the graph and what does it represent? What is the F -intercept and what does it represent?
- 14.** Jason leaves Detroit at 2:00 PM and drives at a constant speed west along I-96. He passes Ann Arbor, 40 mi from Detroit, at 2:50 PM.
- Express the distance traveled in terms of the time elapsed.

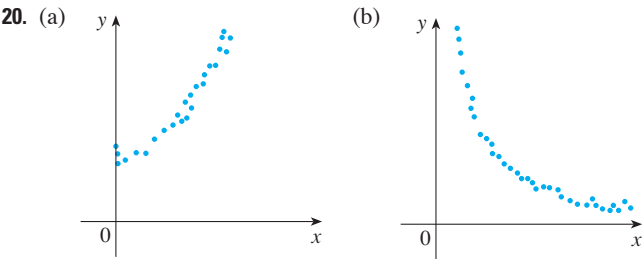
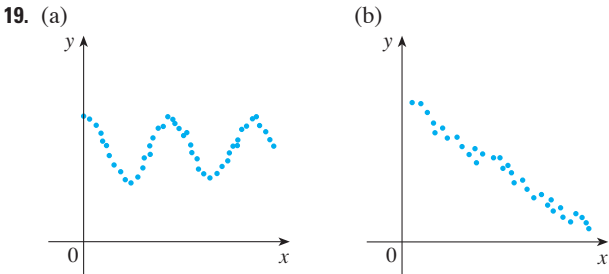


Graphing calculator or computer required

1. Homework Hints available at stewartcalculus.com

- (b) Draw the graph of the equation in part (a).
(c) What is the slope of this line? What does it represent?
15. Biologists have noticed that the chirping rate of crickets of a certain species is related to temperature, and the relationship appears to be very nearly linear. A cricket produces 113 chirps per minute at 70°F and 173 chirps per minute at 80°F.
- (a) Find a linear equation that models the temperature T as a function of the number of chirps per minute N .
(b) What is the slope of the graph? What does it represent?
(c) If the crickets are chirping at 150 chirps per minute, estimate the temperature.
16. The manager of a furniture factory finds that it costs \$2200 to manufacture 100 chairs in one day and \$4800 to produce 300 chairs in one day.
- (a) Express the cost as a function of the number of chairs produced, assuming that it is linear. Then sketch the graph.
(b) What is the slope of the graph and what does it represent?
(c) What is the y -intercept of the graph and what does it represent?
17. At the surface of the ocean, the water pressure is the same as the air pressure above the water, 15 lb/in². Below the surface, the water pressure increases by 4.34 lb/in² for every 10 ft of descent.
- (a) Express the water pressure as a function of the depth below the ocean surface.
(b) At what depth is the pressure 100 lb/in²?
18. The monthly cost of driving a car depends on the number of miles driven. Lynn found that in May it cost her \$380 to drive 480 mi and in June it cost her \$460 to drive 800 mi.
- (a) Express the monthly cost C as a function of the distance driven d , assuming that a linear relationship gives a suitable model.
(b) Use part (a) to predict the cost of driving 1500 miles per month.
(c) Draw the graph of the linear function. What does the slope represent?
(d) What does the C -intercept represent?
(e) Why does a linear function give a suitable model in this situation?

19–20 For each scatter plot, decide what type of function you might choose as a model for the data. Explain your choices.




21. The table shows (lifetime) peptic ulcer rates (per 100 population) for various family incomes as reported by the National Health Interview Survey.

Income	Ulcer rate (per 100 population)
\$4,000	14.1
\$6,000	13.0
\$8,000	13.4
\$12,000	12.5
\$16,000	12.0
\$20,000	12.4
\$30,000	10.5
\$45,000	9.4
\$60,000	8.2

- (a) Make a scatter plot of these data and decide whether a linear model is appropriate.
(b) Find and graph a linear model using the first and last data points.
(c) Find and graph the least squares regression line.
(d) Use the linear model in part (c) to estimate the ulcer rate for an income of \$25,000.
(e) According to the model, how likely is someone with an income of \$80,000 to suffer from peptic ulcers?
(f) Do you think it would be reasonable to apply the model to someone with an income of \$200,000?
22. Biologists have observed that the chirping rate of crickets of a certain species appears to be related to temperature. The table shows the chirping rates for various temperatures.


Temperature (°F)	Chirping rate (chirps/min)	Temperature (°F)	Chirping rate (chirps/min)
50	20	75	140
55	46	80	173
60	79	85	198
65	91	90	211
70	113		

- (a) Make a scatter plot of the data.
(b) Find and graph the regression line.
(c) Use the linear model in part (b) to estimate the chirping rate at 100°F.

-  23. The table gives the winning heights for the men's Olympic pole vault competitions up to the year 2004.

Year	Height (m)	Year	Height (m)
1896	3.30	1960	4.70
1900	3.30	1964	5.10
1904	3.50	1968	5.40
1908	3.71	1972	5.64
1912	3.95	1976	5.64
1920	4.09	1980	5.78
1924	3.95	1984	5.75
1928	4.20	1988	5.90
1932	4.31	1992	5.87
1936	4.35	1996	5.92
1948	4.30	2000	5.90
1952	4.55	2004	5.95
1956	4.56		

- Make a scatter plot and decide whether a linear model is appropriate.
- Find and graph the regression line.
- Use the linear model to predict the height of the winning pole vault at the 2008 Olympics and compare with the actual winning height of 5.96 meters.
- Is it reasonable to use the model to predict the winning height at the 2100 Olympics?


-  24. The table shows the percentage of the population of Argentina that has lived in rural areas from 1955 to 2000. Find a model for the data and use it to estimate the rural percentage in 1988 and 2002.

Year	Percentage rural	Year	Percentage rural
1955	30.4	1980	17.1
1960	26.4	1985	15.0
1965	23.6	1990	13.0
1970	21.1	1995	11.7
1975	19.0	2000	10.5

- Many physical quantities are connected by *inverse square laws*, that is, by power functions of the form $f(x) = kx^{-2}$. In particular, the illumination of an object by a light source is inversely proportional to the square of the distance from the source. Suppose that after dark you are in a room with just one lamp and you are trying to read a book. The light is too dim and so you move halfway to the lamp. How much brighter is the light?
- It makes sense that the larger the area of a region, the larger the number of species that inhabit the region. Many


ecologists have modeled the species-area relation with a power function and, in particular, the number of species S of bats living in caves in central Mexico has been related to the surface area A of the caves by the equation $S = 0.7A^{0.3}$.

- The cave called *Misión Imposible* near Puebla, Mexico, has a surface area of $A = 60 \text{ m}^2$. How many species of bats would you expect to find in that cave?
- If you discover that four species of bats live in a cave, estimate the area of the cave.

-  27. The table shows the number N of species of reptiles and amphibians inhabiting Caribbean islands and the area A of the island in square miles.

Island	A	N
Saba	4	5
Montserrat	40	9
Puerto Rico	3,459	40
Jamaica	4,411	39
Hispaniola	29,418	84
Cuba	44,218	76

- Use a power function to model N as a function of A .
- The Caribbean island of Dominica has area 291 m^2 . How many species of reptiles and amphibians would you expect to find on Dominica?

-  28. The table shows the mean (average) distances d of the planets from the sun (taking the unit of measurement to be the distance from the earth to the sun) and their periods T (time of revolution in years).

Planet	d	T
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.523	1.881
Jupiter	5.203	11.861
Saturn	9.541	29.457
Uranus	19.190	84.008
Neptune	30.086	164.784

- Fit a power model to the data.
- Kepler's Third Law of Planetary Motion states that "The square of the period of revolution of a planet is proportional to the cube of its mean distance from the sun."

Does your model corroborate Kepler's Third Law?

1.3 New Functions from Old Functions

In this section we start with the basic functions we discussed in Section 1.2 and obtain new functions by shifting, stretching, and reflecting their graphs. We also show how to combine pairs of functions by the standard arithmetic operations and by composition.

Transformations of Functions

By applying certain transformations to the graph of a given function we can obtain the graphs of certain related functions. This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given graphs. Let's first consider **translations**. If c is a positive number, then the graph of $y = f(x) + c$ is just the graph of $y = f(x)$ shifted upward a distance of c units (because each y -coordinate is increased by the same number c). Likewise, if $g(x) = f(x - c)$, where $c > 0$, then the value of g at x is the same as the value of f at $x - c$ (c units to the left of x). Therefore the graph of $y = f(x - c)$ is just the graph of $y = f(x)$ shifted c units to the right (see Figure 1).

Vertical and Horizontal Shifts Suppose $c > 0$. To obtain the graph of
 $y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
 $y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
 $y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
 $y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

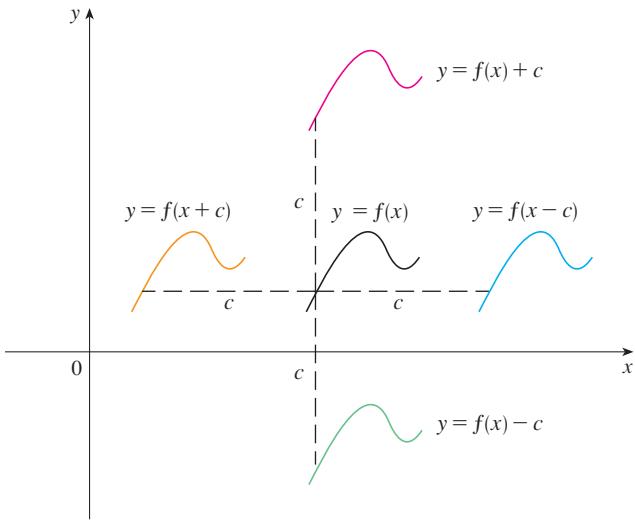


FIGURE 1
Translating the graph of f

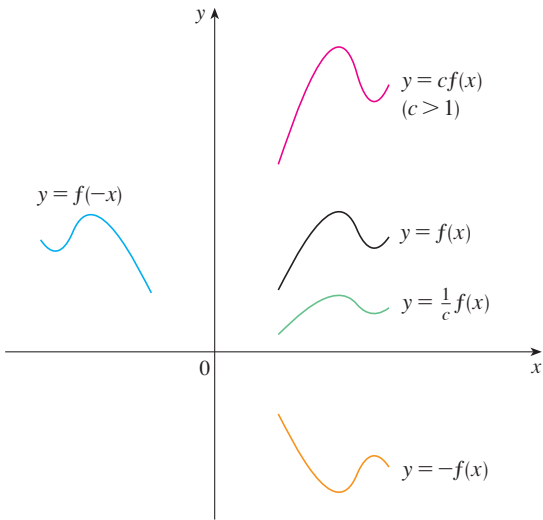


FIGURE 2
Stretching and reflecting the graph of f

Now let's consider the **stretching** and **reflecting** transformations. If $c > 1$, then the graph of $y = cf(x)$ is the graph of $y = f(x)$ stretched by a factor of c in the vertical direction (because each y -coordinate is multiplied by the same number c). The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x -axis because the point (x, y) is

replaced by the point $(x, -y)$. (See Figure 2 and the following chart, where the results of other stretching, shrinking, and reflecting transformations are also given.)

Vertical and Horizontal Stretching and Reflecting Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c

$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c

$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c

$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c

$y = -f(x)$, reflect the graph of $y = f(x)$ about the x -axis

$y = f(-x)$, reflect the graph of $y = f(x)$ about the y -axis

Figure 3 illustrates these stretching transformations when applied to the cosine function with $c = 2$. For instance, in order to get the graph of $y = 2 \cos x$ we multiply the y -coordinate of each point on the graph of $y = \cos x$ by 2. This means that the graph of $y = \cos x$ gets stretched vertically by a factor of 2.

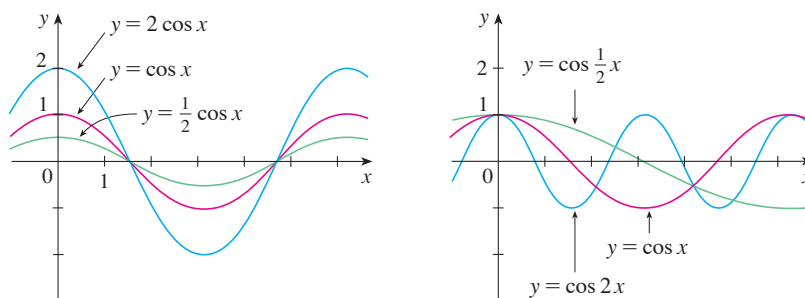


FIGURE 3

EXAMPLE 1 Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{x} - 2$, $y = \sqrt{x - 2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.

SOLUTION The graph of the square root function $y = \sqrt{x}$, obtained from Figure 13(a) in Section 1.2, is shown in Figure 4(a). In the other parts of the figure we sketch $y = \sqrt{x} - 2$ by shifting 2 units downward, $y = \sqrt{x - 2}$ by shifting 2 units to the right, $y = -\sqrt{x}$ by reflecting about the x -axis, $y = 2\sqrt{x}$ by stretching vertically by a factor of 2, and $y = \sqrt{-x}$ by reflecting about the y -axis.

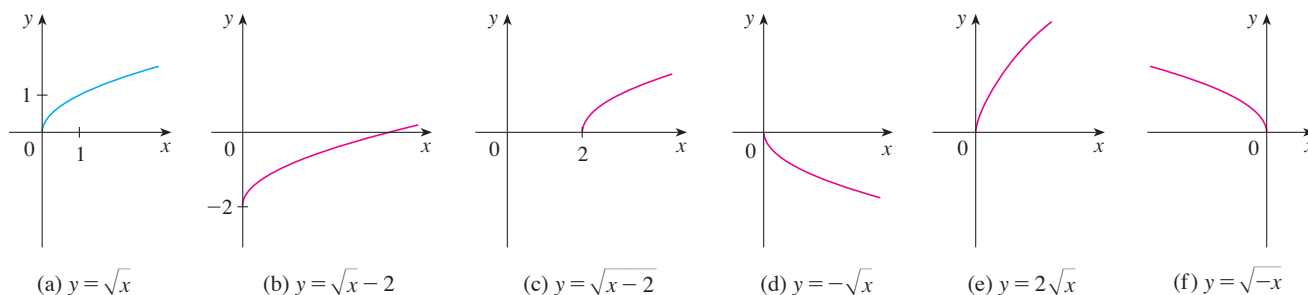


FIGURE 4

EXAMPLE 2 Sketch the graph of the function $f(x) = x^2 + 6x + 10$.

SOLUTION Completing the square, we write the equation of the graph as

$$y = x^2 + 6x + 10 = (x + 3)^2 + 1$$

This means we obtain the desired graph by starting with the parabola $y = x^2$ and shifting 3 units to the left and then 1 unit upward (see Figure 5).

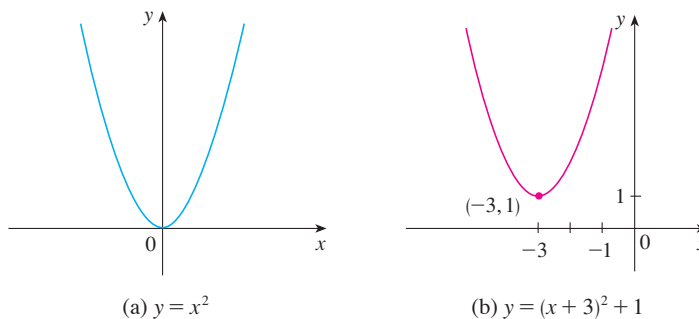


FIGURE 5

EXAMPLE 3 Sketch the graphs of the following functions.

(a) $y = \sin 2x$

(b) $y = 1 - \sin x$

SOLUTION

(a) We obtain the graph of $y = \sin 2x$ from that of $y = \sin x$ by compressing horizontally by a factor of 2. (See Figures 6 and 7.) Thus, whereas the period of $y = \sin x$ is 2π , the period of $y = \sin 2x$ is $2\pi/2 = \pi$.

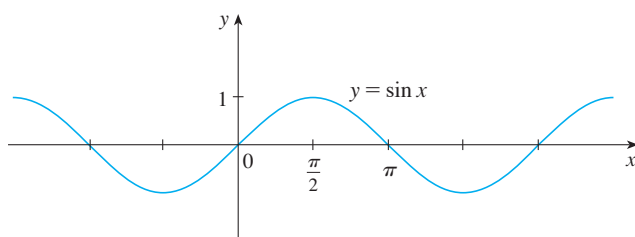


FIGURE 6

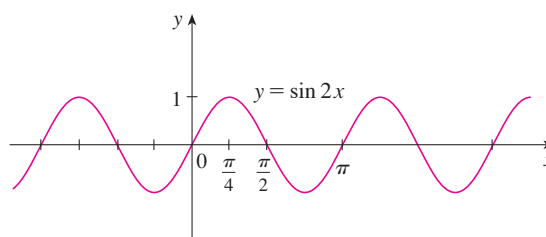


FIGURE 7

(b) To obtain the graph of $y = 1 - \sin x$, we again start with $y = \sin x$. We reflect about the x -axis to get the graph of $y = -\sin x$ and then we shift 1 unit upward to get $y = 1 - \sin x$. (See Figure 8.)

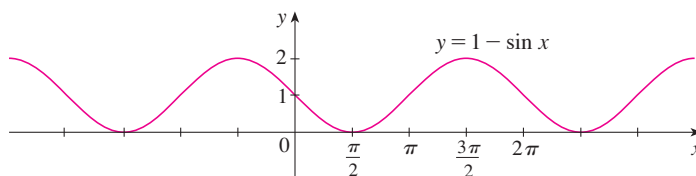
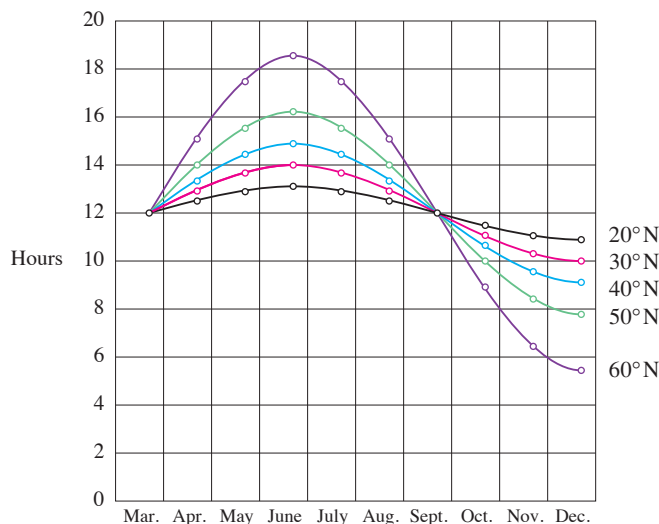


FIGURE 8

EXAMPLE 4 Figure 9 shows graphs of the number of hours of daylight as functions of the time of the year at several latitudes. Given that Philadelphia is located at approximately 40°N latitude, find a function that models the length of daylight at Philadelphia.

FIGURE 9
Graph of the length of daylight
from March 21 through December 21
at various latitudes

Lucia C. Harrison, *Daylight, Twilight, Darkness and Time*
(New York, 1935) page 40.



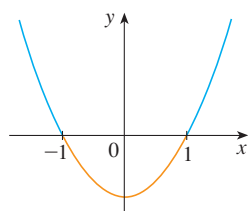
SOLUTION Notice that each curve resembles a shifted and stretched sine function. By looking at the blue curve we see that, at the latitude of Philadelphia, daylight lasts about 14.8 hours on June 21 and 9.2 hours on December 21, so the amplitude of the curve (the factor by which we have to stretch the sine curve vertically) is $\frac{1}{2}(14.8 - 9.2) = 2.8$.

By what factor do we need to stretch the sine curve horizontally if we measure the time t in days? Because there are about 365 days in a year, the period of our model should be 365. But the period of $y = \sin t$ is 2π , so the horizontal stretching factor is $c = 2\pi/365$.

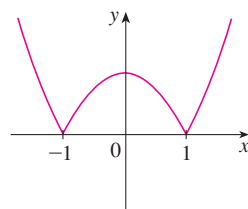
We also notice that the curve begins its cycle on March 21, the 80th day of the year, so we have to shift the curve 80 units to the right. In addition, we shift it 12 units upward. Therefore we model the length of daylight in Philadelphia on the t th day of the year by the function

$$L(t) = 12 + 2.8 \sin \left[\frac{2\pi}{365}(t - 80) \right]$$

Another transformation of some interest is taking the *absolute value* of a function. If $y = |f(x)|$, then according to the definition of absolute value, $y = f(x)$ when $f(x) \geq 0$ and $y = -f(x)$ when $f(x) < 0$. This tells us how to get the graph of $y = |f(x)|$ from the graph of $y = f(x)$: The part of the graph that lies above the x -axis remains the same; the part that lies below the x -axis is reflected about the x -axis.



(a) $y = x^2 - 1$



(b) $y = |x^2 - 1|$

V EXAMPLE 5 Sketch the graph of the function $y = |x^2 - 1|$.

SOLUTION We first graph the parabola $y = x^2 - 1$ in Figure 10(a) by shifting the parabola $y = x^2$ downward 1 unit. We see that the graph lies below the x -axis when $-1 < x < 1$, so we reflect that part of the graph about the x -axis to obtain the graph of $y = |x^2 - 1|$ in Figure 10(b).

Combinations of Functions

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg , and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers. The sum and difference functions are defined by

$$(f + g)(x) = f(x) + g(x) \quad (f - g)(x) = f(x) - g(x)$$

FIGURE 10

If the domain of f is A and the domain of g is B , then the domain of $f + g$ is the intersection $A \cap B$ because both $f(x)$ and $g(x)$ have to be defined. For example, the domain of $f(x) = \sqrt{x}$ is $A = [0, \infty)$ and the domain of $g(x) = \sqrt{2 - x}$ is $B = (-\infty, 2]$, so the domain of $(f + g)(x) = \sqrt{x} + \sqrt{2 - x}$ is $A \cap B = [0, 2]$.

Similarly, the product and quotient functions are defined by

$$(fg)(x) = f(x)g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The domain of fg is $A \cap B$, but we can't divide by 0 and so the domain of f/g is $\{x \in A \cap B \mid g(x) \neq 0\}$. For instance, if $f(x) = x^2$ and $g(x) = x - 1$, then the domain of the rational function $(f/g)(x) = x^2/(x - 1)$ is $\{x \mid x \neq 1\}$, or $(-\infty, 1) \cup (1, \infty)$.

There is another way of combining two functions to obtain a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$. Since y is a function of u and u is, in turn, a function of x , it follows that y is ultimately a function of x . We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions f and g .

In general, given any two functions f and g , we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f , then we can calculate the value of $f(g(x))$. Notice that the output of one function is used as the input to the next function. The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f . It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ (“ f circle g ”).

Definition Given two functions f and g , the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

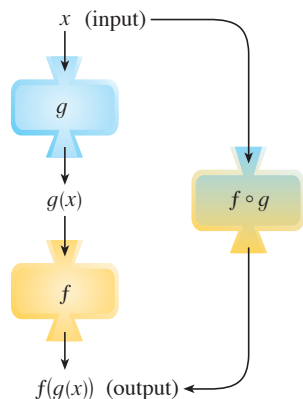


FIGURE 11

The $f \circ g$ machine is composed of the g machine (first) and then the f machine.

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f . In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined. Figure 11 shows how to picture $f \circ g$ in terms of machines.

EXAMPLE 6 If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

SOLUTION We have

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$$

NOTE You can see from Example 6 that, in general, $f \circ g \neq g \circ f$. Remember, the notation $f \circ g$ means that the function g is applied first and then f is applied second. In Example 6, $f \circ g$ is the function that *first* subtracts 3 and *then* squares; $g \circ f$ is the function that *first* squares and *then* subtracts 3.

V EXAMPLE 7 If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$, find each function and its domain.

- (a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

SOLUTION

$$(a) \quad (f \circ g)(x) = f(g(x)) = f(\sqrt{2-x}) = \sqrt{\sqrt{2-x}} = \sqrt[4]{2-x}$$

The domain of $f \circ g$ is $\{x \mid 2-x \geq 0\} = \{x \mid x \leq 2\} = (-\infty, 2]$.

$$(b) \quad (g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{2-\sqrt{x}}$$

If $0 \leq a \leq b$, then $a^2 \leq b^2$.

For \sqrt{x} to be defined we must have $x \geq 0$. For $\sqrt{2-\sqrt{x}}$ to be defined we must have $2-\sqrt{x} \geq 0$, that is, $\sqrt{x} \leq 2$, or $x \leq 4$. Thus we have $0 \leq x \leq 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.

$$(c) \quad (f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$$

The domain of $f \circ f$ is $[0, \infty)$.

$$(d) \quad (g \circ g)(x) = g(g(x)) = g(\sqrt{2-x}) = \sqrt{2-\sqrt{2-x}}$$

This expression is defined when both $2-x \geq 0$ and $2-\sqrt{2-x} \geq 0$. The first inequality means $x \leq 2$, and the second is equivalent to $\sqrt{2-x} \leq 2$, or $2-x \leq 4$, or $x \geq -2$. Thus $-2 \leq x \leq 2$, so the domain of $g \circ g$ is the closed interval $[-2, 2]$.

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h , then g , and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

EXAMPLE 8 Find $f \circ g \circ h$ if $f(x) = x/(x+1)$, $g(x) = x^{10}$, and $h(x) = x+3$.

SOLUTION

$$\begin{aligned} (f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x+3)) \\ &= f((x+3)^{10}) = \frac{(x+3)^{10}}{(x+3)^{10}+1} \end{aligned}$$

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to *decompose* a complicated function into simpler ones, as in the following example.

EXAMPLE 9 Given $F(x) = \cos^2(x+9)$, find functions f , g , and h such that $F = f \circ g \circ h$.

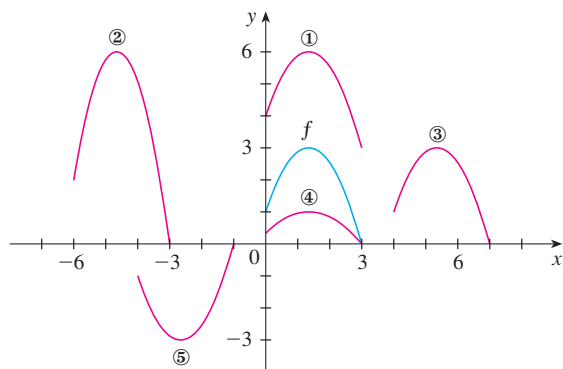
SOLUTION Since $F(x) = [\cos(x+9)]^2$, the formula for F says: First add 9, then take the cosine of the result, and finally square. So we let

$$h(x) = x+9 \quad g(x) = \cos x \quad f(x) = x^2$$

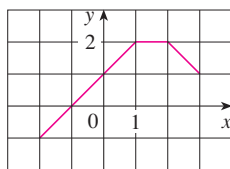
$$\begin{aligned} \text{Then} \quad (f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x+9)) = f(\cos(x+9)) \\ &= [\cos(x+9)]^2 = F(x) \end{aligned}$$

1.3 Exercises

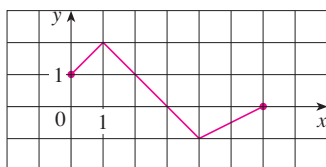
1. Suppose the graph of f is given. Write equations for the graphs that are obtained from the graph of f as follows.
- (a) Shift 3 units upward. (b) Shift 3 units downward.
 (c) Shift 3 units to the right. (d) Shift 3 units to the left.
 (e) Reflect about the x -axis. (f) Reflect about the y -axis.
 (g) Stretch vertically by a factor of 3.
 (h) Shrink vertically by a factor of 3.
2. Explain how each graph is obtained from the graph of $y = f(x)$.
- (a) $y = f(x) + 8$ (b) $y = f(x + 8)$
 (c) $y = 8f(x)$ (d) $y = f(8x)$
 (e) $y = -f(x) - 1$ (f) $y = 8f(\frac{1}{8}x)$
3. The graph of $y = f(x)$ is given. Match each equation with its graph and give reasons for your choices.
- (a) $y = f(x - 4)$ (b) $y = f(x) + 3$
 (c) $y = \frac{1}{3}f(x)$ (d) $y = -f(x + 4)$
 (e) $y = 2f(x + 6)$



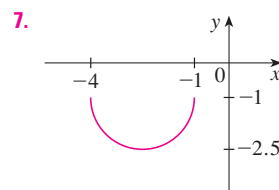
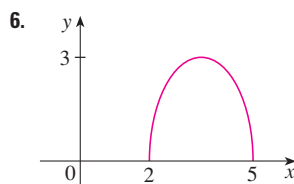
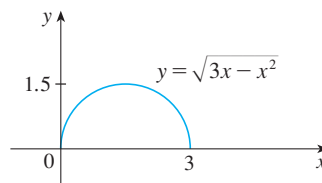
4. The graph of f is given. Draw the graphs of the following functions.
- (a) $y = f(x) - 2$ (b) $y = f(x - 2)$
 (c) $y = -2f(x)$ (d) $y = f(\frac{1}{3}x) + 1$



5. The graph of f is given. Use it to graph the following functions.
- (a) $y = f(2x)$ (b) $y = f(\frac{1}{2}x)$
 (c) $y = f(-x)$ (d) $y = -f(-x)$



- 6–7 The graph of $y = \sqrt{3x - x^2}$ is given. Use transformations to create a function whose graph is as shown.



8. (a) How is the graph of $y = 2 \sin x$ related to the graph of $y = \sin x$? Use your answer and Figure 6 to sketch the graph of $y = 2 \sin x$.
 (b) How is the graph of $y = 1 + \sqrt{x}$ related to the graph of $y = \sqrt{x}$? Use your answer and Figure 4(a) to sketch the graph of $y = 1 + \sqrt{x}$.

9–24 Graph the function by hand, not by plotting points, but by starting with the graph of one of the standard functions given in Section 1.2, and then applying the appropriate transformations.

9. $y = \frac{1}{x+2}$

10. $y = (x-1)^3$

11. $y = -\sqrt[3]{x}$

12. $y = x^2 + 6x + 4$

13. $y = \sqrt{x-2} - 1$

14. $y = 4 \sin 3x$

15. $y = \sin(\frac{1}{2}x)$

16. $y = \frac{2}{x} - 2$

17. $y = \frac{1}{2}(1 - \cos x)$

18. $y = 1 - 2\sqrt{x+3}$

19. $y = 1 - 2x - x^2$

20. $y = |x| - 2$

21. $y = |x-2|$

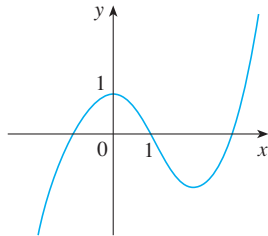
22. $y = \frac{1}{4} \tan\left(x - \frac{\pi}{4}\right)$

23. $y = |\sqrt{x} - 1|$

24. $y = |\cos \pi x|$

25. The city of New Orleans is located at latitude 30°N . Use Figure 9 to find a function that models the number of hours of daylight at New Orleans as a function of the time of year. To check the accuracy of your model, use the fact that on March 31 the sun rises at 5:51 AM and sets at 6:18 PM in New Orleans.

26. A variable star is one whose brightness alternately increases and decreases. For the most visible variable star, Delta Cephei, the time between periods of maximum brightness is 5.4 days, the average brightness (or magnitude) of the star is 4.0, and its brightness varies by ± 0.35 magnitude. Find a function that models the brightness of Delta Cephei as a function of time.
27. (a) How is the graph of $y = f(|x|)$ related to the graph of f ?
 (b) Sketch the graph of $y = \sin |x|$.
 (c) Sketch the graph of $y = \sqrt{|x|}$.
28. Use the given graph of f to sketch the graph of $y = 1/f(x)$. Which features of f are the most important in sketching $y = 1/f(x)$? Explain how they are used.



29–30 Find (a) $f + g$, (b) $f - g$, (c) fg , and (d) f/g and state their domains.

29. $f(x) = x^3 + 2x^2$, $g(x) = 3x^2 - 1$

30. $f(x) = \sqrt{3-x}$, $g(x) = \sqrt{x^2-1}$

31–36 Find the functions (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, and (d) $g \circ g$ and their domains.

31. $f(x) = x^2 - 1$, $g(x) = 2x + 1$

32. $f(x) = x - 2$, $g(x) = x^2 + 3x + 4$

33. $f(x) = 1 - 3x$, $g(x) = \cos x$

34. $f(x) = \sqrt{x}$, $g(x) = \sqrt[3]{1-x}$

35. $f(x) = x + \frac{1}{x}$, $g(x) = \frac{x+1}{x+2}$

36. $f(x) = \frac{x}{1+x}$, $g(x) = \sin 2x$

37–40 Find $f \circ g \circ h$.

37. $f(x) = 3x - 2$, $g(x) = \sin x$, $h(x) = x^2$

38. $f(x) = |x - 4|$, $g(x) = 2^x$, $h(x) = \sqrt{x}$

39. $f(x) = \sqrt{x-3}$, $g(x) = x^2$, $h(x) = x^3 + 2$

40. $f(x) = \tan x$, $g(x) = \frac{x}{x-1}$, $h(x) = \sqrt[3]{x}$

41–46 Express the function in the form $f \circ g$.

41. $F(x) = (2x + x^2)^4$

42. $F(x) = \cos^2 x$

43. $F(x) = \frac{\sqrt[3]{x}}{1 + \sqrt[3]{x}}$

44. $G(x) = \sqrt[3]{\frac{x}{1+x}}$

45. $v(t) = \sec(t^2) \tan(t^2)$

46. $u(t) = \frac{\tan t}{1 + \tan t}$

47–49 Express the function in the form $f \circ g \circ h$.

47. $R(x) = \sqrt{\sqrt{x} - 1}$

48. $H(x) = \sqrt[8]{2 + |x|}$

49. $H(x) = \sec^4(\sqrt{x})$

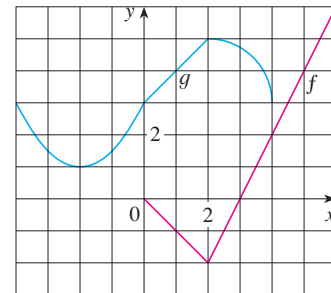
50. Use the table to evaluate each expression.

- (a) $f(g(1))$ (b) $g(f(1))$ (c) $f(f(1))$
 (d) $g(g(1))$ (e) $(g \circ f)(3)$ (f) $(f \circ g)(6)$

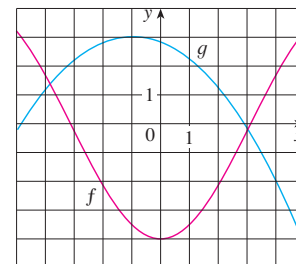
x	1	2	3	4	5	6
$f(x)$	3	1	4	2	2	5
$g(x)$	6	3	2	1	2	3

51. Use the given graphs of f and g to evaluate each expression, or explain why it is undefined.

- (a) $f(g(2))$ (b) $g(f(0))$ (c) $(f \circ g)(0)$
 (d) $(g \circ f)(6)$ (e) $(g \circ g)(-2)$ (f) $(f \circ f)(4)$



52. Use the given graphs of f and g to estimate the value of $f(g(x))$ for $x = -5, -4, -3, \dots, 5$. Use these estimates to sketch a rough graph of $f \circ g$.



53. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.
- Express the radius r of this circle as a function of the time t (in seconds).
 - If A is the area of this circle as a function of the radius, find $A \circ r$ and interpret it.
54. A spherical balloon is being inflated and the radius of the balloon is increasing at a rate of 2 cm/s.
- Express the radius r of the balloon as a function of the time t (in seconds).
 - If V is the volume of the balloon as a function of the radius, find $V \circ r$ and interpret it.
55. A ship is moving at a speed of 30 km/h parallel to a straight shoreline. The ship is 6 km from shore and it passes a lighthouse at noon.
- Express the distance s between the lighthouse and the ship as a function of d , the distance the ship has traveled since noon; that is, find f so that $s = f(d)$.
 - Express d as a function of t , the time elapsed since noon; that is, find g so that $d = g(t)$.
 - Find $f \circ g$. What does this function represent?
56. An airplane is flying at a speed of 350 mi/h at an altitude of one mile and passes directly over a radar station at time $t = 0$.
- Express the horizontal distance d (in miles) that the plane has flown as a function of t .
 - Express the distance s between the plane and the radar station as a function of d .
 - Use composition to express s as a function of t .

57. The **Heaviside function** H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.

- Sketch the graph of the Heaviside function.
- Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and 120 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.

- Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 5$ seconds and 240 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$. (Note that starting at $t = 5$ corresponds to a translation.)

58. The Heaviside function defined in Exercise 57 can also be used to define the **ramp function** $y = ctH(t)$, which represents a gradual increase in voltage or current in a circuit.

- Sketch the graph of the ramp function $y = tH(t)$.
- Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and the voltage is gradually increased to 120 volts over a 60-second time interval. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 60$.
- Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 7$ seconds and the voltage is gradually increased to 100 volts over a period of 25 seconds. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 32$.

59. Let f and g be linear functions with equations $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$. Is $f \circ g$ also a linear function? If so, what is the slope of its graph?

60. If you invest x dollars at 4% interest compounded annually, then the amount $A(x)$ of the investment after one year is $A(x) = 1.04x$. Find $A \circ A$, $A \circ A \circ A$, and $A \circ A \circ A \circ A$. What do these compositions represent? Find a formula for the composition of n copies of A .

61. (a) If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$. (Think about what operations you would have to perform on the formula for g to end up with the formula for h .)
- (b) If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

62. If $f(x) = x + 4$ and $h(x) = 4x - 1$, find a function g such that $g \circ f = h$.

63. Suppose g is an even function and let $h = f \circ g$. Is h always an even function?

64. Suppose g is an odd function and let $h = f \circ g$. Is h always an odd function? What if f is odd? What if f is even?

1.4 Graphing Calculators and Computers

In this section we assume that you have access to a graphing calculator or a computer with graphing software. We will see that the use of such a device enables us to graph more complicated functions and to solve more complex problems than would otherwise be possible. We also point out some of the pitfalls that can occur with these machines.

Graphing calculators and computers can give very accurate graphs of functions. But we will see in Chapter 4 that only through the use of calculus can we be sure that we have uncovered all the interesting aspects of a graph.

A graphing calculator or computer displays a rectangular portion of the graph of a function in a **display window** or **viewing screen**, which we refer to as a **viewing rectangle**. The default screen often gives an incomplete or misleading picture, so it is important to

choose the viewing rectangle with care. If we choose the x -values to range from a minimum value of $Xmin = a$ to a maximum value of $Xmax = b$ and the y -values to range from a minimum of $Ymin = c$ to a maximum of $Ymax = d$, then the visible portion of the graph lies in the rectangle

$$[a, b] \times [c, d] = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$$

shown in Figure 1. We refer to this rectangle as the $[a, b]$ by $[c, d]$ viewing rectangle.

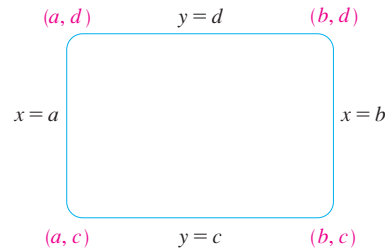


FIGURE 1

The viewing rectangle $[a, b]$ by $[c, d]$

The machine draws the graph of a function f much as you would. It plots points of the form $(x, f(x))$ for a certain number of equally spaced values of x between a and b . If an x -value is not in the domain of f , or if $f(x)$ lies outside the viewing rectangle, it moves on to the next x -value. The machine connects each point to the preceding plotted point to form a representation of the graph of f .

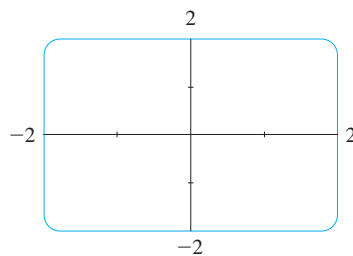
EXAMPLE 1 Draw the graph of the function $f(x) = x^2 + 3$ in each of the following viewing rectangles.

(a) $[-2, 2]$ by $[-2, 2]$

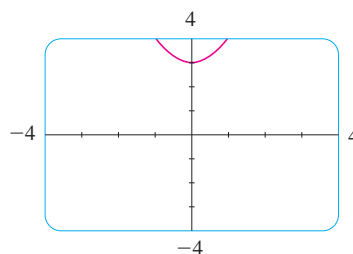
(b) $[-4, 4]$ by $[-4, 4]$

(c) $[-10, 10]$ by $[-5, 30]$

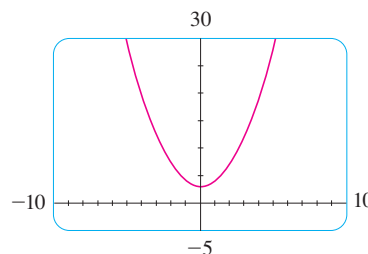
(d) $[-50, 50]$ by $[-100, 1000]$



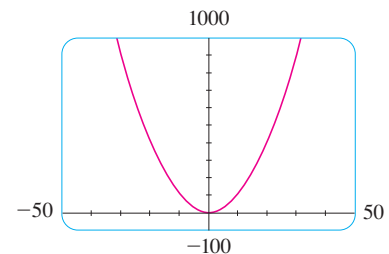
(a) $[-2, 2]$ by $[-2, 2]$



(b) $[-4, 4]$ by $[-4, 4]$



(c) $[-10, 10]$ by $[-5, 30]$



(d) $[-50, 50]$ by $[-100, 1000]$

FIGURE 2 Graphs of $f(x) = x^2 + 3$

We see from Example 1 that the choice of a viewing rectangle can make a big difference in the appearance of a graph. Often it's necessary to change to a larger viewing rectangle to obtain a more complete picture, a more global view, of the graph. In the next example we see that knowledge of the domain and range of a function sometimes provides us with enough information to select a good viewing rectangle.

EXAMPLE 2 Determine an appropriate viewing rectangle for the function $f(x) = \sqrt{8 - 2x^2}$ and use it to graph f .

SOLUTION The expression for $f(x)$ is defined when

$$\begin{aligned} 8 - 2x^2 \geq 0 &\iff 2x^2 \leq 8 \iff x^2 \leq 4 \\ &\iff |x| \leq 2 \iff -2 \leq x \leq 2 \end{aligned}$$

Therefore the domain of f is the interval $[-2, 2]$. Also,

$$0 \leq \sqrt{8 - 2x^2} \leq \sqrt{8} = 2\sqrt{2} \approx 2.83$$

so the range of f is the interval $[0, 2\sqrt{2}]$.

We choose the viewing rectangle so that the x -interval is somewhat larger than the domain and the y -interval is larger than the range. Taking the viewing rectangle to be $[-3, 3]$ by $[-1, 4]$, we get the graph shown in Figure 3.

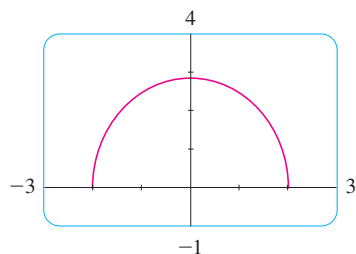


FIGURE 3
 $f(x) = \sqrt{8 - 2x^2}$

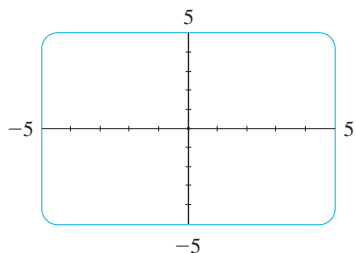


FIGURE 4

EXAMPLE 3 Graph the function $y = x^3 - 150x$.

SOLUTION Here the domain is \mathbb{R} , the set of all real numbers. That doesn't help us choose a viewing rectangle. Let's experiment. If we start with the viewing rectangle $[-5, 5]$ by $[-5, 5]$, we get the graph in Figure 4. It appears blank, but actually the graph is so nearly vertical that it blends in with the y -axis.

If we change the viewing rectangle to $[-20, 20]$ by $[-20, 20]$, we get the picture shown in Figure 5(a). The graph appears to consist of vertical lines, but we know that can't be correct. If we look carefully while the graph is being drawn, we see that the graph leaves the screen and reappears during the graphing process. This indicates that we need to see more in the vertical direction, so we change the viewing rectangle to $[-20, 20]$ by $[-500, 500]$. The resulting graph is shown in Figure 5(b). It still doesn't quite reveal all the main features of the function, so we try $[-20, 20]$ by $[-1000, 1000]$ in Figure 5(c). Now we are more confident that we have arrived at an appropriate viewing rectangle. In Chapter 4 we will be able to see that the graph shown in Figure 5(c) does indeed reveal all the main features of the function.

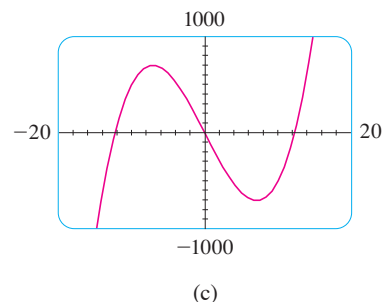
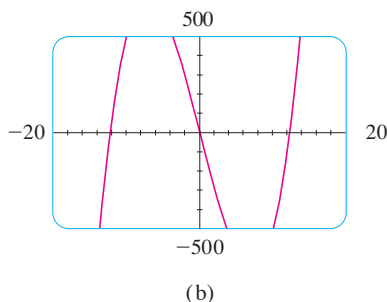
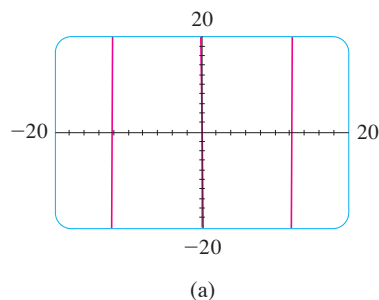
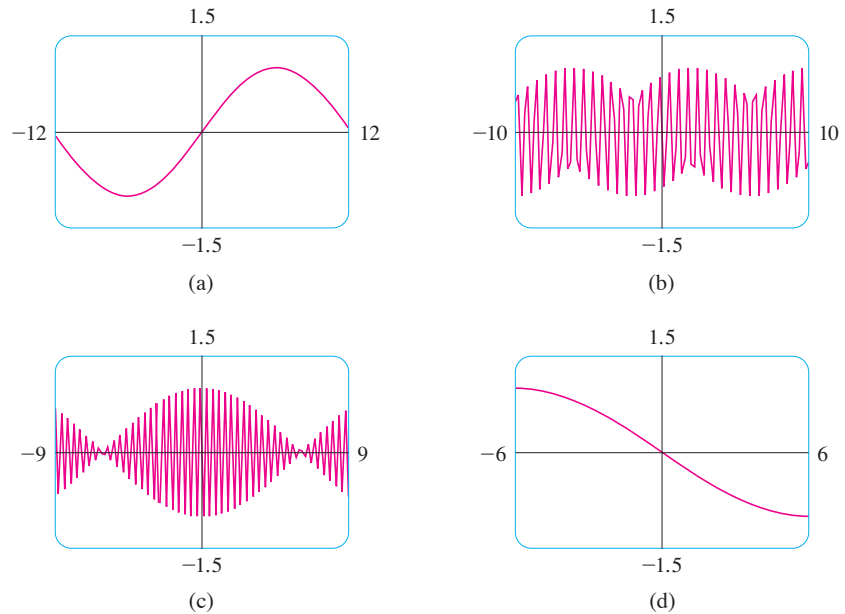


FIGURE 5 Graphs of $y = x^3 - 150x$

V EXAMPLE 4 Graph the function $f(x) = \sin 50x$ in an appropriate viewing rectangle.

SOLUTION Figure 6(a) shows the graph of f produced by a graphing calculator using the viewing rectangle $[-12, 12]$ by $[-1.5, 1.5]$. At first glance the graph appears to be

reasonable. But if we change the viewing rectangle to the ones shown in the following parts of Figure 6, the graphs look very different. Something strange is happening.



The appearance of the graphs in Figure 6 depends on the machine used. The graphs you get with your own graphing device might not look like these figures, but they will also be quite inaccurate.

FIGURE 6
Graphs of $f(x) = \sin 50x$
in four viewing rectangles

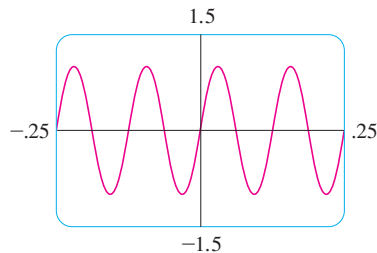


FIGURE 7
 $f(x) = \sin 50x$

In order to explain the big differences in appearance of these graphs and to find an appropriate viewing rectangle, we need to find the period of the function $y = \sin 50x$. We know that the function $y = \sin x$ has period 2π and the graph of $y = \sin 50x$ is shrunk horizontally by a factor of 50, so the period of $y = \sin 50x$ is

$$\frac{2\pi}{50} = \frac{\pi}{25} \approx 0.126$$

This suggests that we should deal only with small values of x in order to show just a few oscillations of the graph. If we choose the viewing rectangle $[-0.25, 0.25]$ by $[-1.5, 1.5]$, we get the graph shown in Figure 7.

Now we see what went wrong in Figure 6. The oscillations of $y = \sin 50x$ are so rapid that when the calculator plots points and joins them, it misses most of the maximum and minimum points and therefore gives a very misleading impression of the graph.

We have seen that the use of an inappropriate viewing rectangle can give a misleading impression of the graph of a function. In Examples 1 and 3 we solved the problem by changing to a larger viewing rectangle. In Example 4 we had to make the viewing rectangle smaller. In the next example we look at a function for which there is no single viewing rectangle that reveals the true shape of the graph.

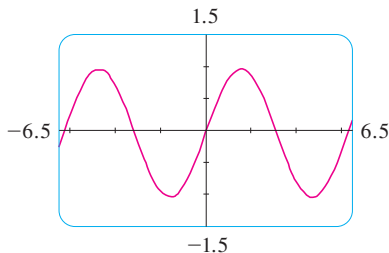


FIGURE 8

V EXAMPLE 5 Graph the function $f(x) = \sin x + \frac{1}{100} \cos 100x$.

SOLUTION Figure 8 shows the graph of f produced by a graphing calculator with viewing rectangle $[-6.5, 6.5]$ by $[-1.5, 1.5]$. It looks much like the graph of $y = \sin x$, but perhaps with some bumps attached. If we zoom in to the viewing rectangle $[-0.1, 0.1]$ by

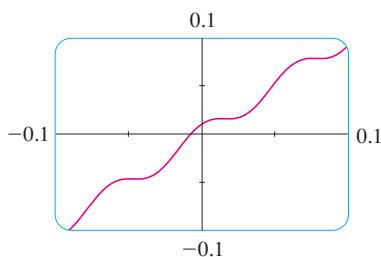


FIGURE 9

Another way to avoid the extraneous line is to change the graphing mode on the calculator so that the dots are not connected.

$[-0.1, 0.1]$, we can see much more clearly the shape of these bumps in Figure 9. The reason for this behavior is that the second term, $\frac{1}{100} \cos 100x$, is very small in comparison with the first term, $\sin x$. Thus we really need two graphs to see the true nature of this function.

EXAMPLE 6 Draw the graph of the function $y = \frac{1}{1-x}$.

SOLUTION Figure 10(a) shows the graph produced by a graphing calculator with viewing rectangle $[-9, 9]$ by $[-9, 9]$. In connecting successive points on the graph, the calculator produced a steep line segment from the top to the bottom of the screen. That line segment is not truly part of the graph. Notice that the domain of the function $y = 1/(1-x)$ is $\{x \mid x \neq 1\}$. We can eliminate the extraneous near-vertical line by experimenting with a change of scale. When we change to the smaller viewing rectangle $[-4.7, 4.7]$ by $[-4.7, 4.7]$ on this particular calculator, we obtain the much better graph in Figure 10(b).

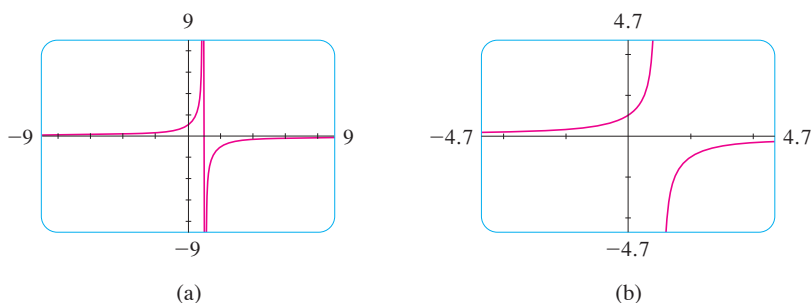


FIGURE 10

EXAMPLE 7 Graph the function $y = \sqrt[3]{x}$.

SOLUTION Some graphing devices display the graph shown in Figure 11, whereas others produce a graph like that in Figure 12. We know from Section 1.2 (Figure 13) that the graph in Figure 12 is correct, so what happened in Figure 11? The explanation is that some machines compute the cube root of x using a logarithm, which is not defined if x is negative, so only the right half of the graph is produced.

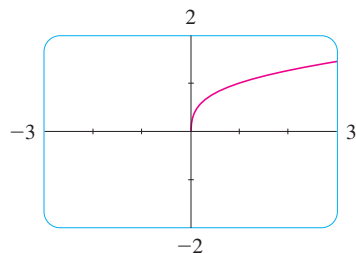


FIGURE 11

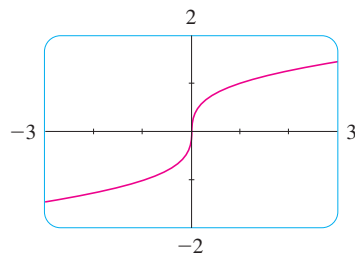


FIGURE 12

You can get the correct graph with Maple if you first type

```
with(RealDomain);
```

You should experiment with your own machine to see which of these two graphs is produced. If you get the graph in Figure 11, you can obtain the correct picture by graphing the function

$$f(x) = \frac{x}{|x|} \cdot |x|^{1/3}$$

Notice that this function is equal to $\sqrt[3]{x}$ (except when $x = 0$).

To understand how the expression for a function relates to its graph, it's helpful to graph a **family of functions**, that is, a collection of functions whose equations are related. In the next example we graph members of a family of cubic polynomials.

V EXAMPLE 8 Graph the function $y = x^3 + cx$ for various values of the number c . How does the graph change when c is changed?

TEC In Visual 1.4 you can see an animation of Figure 13.

SOLUTION Figure 13 shows the graphs of $y = x^3 + cx$ for $c = 2, 1, 0, -1$, and -2 . We see that, for positive values of c , the graph increases from left to right with no maximum or minimum points (peaks or valleys). When $c = 0$, the curve is flat at the origin. When c is negative, the curve has a maximum point and a minimum point. As c decreases, the maximum point becomes higher and the minimum point lower.

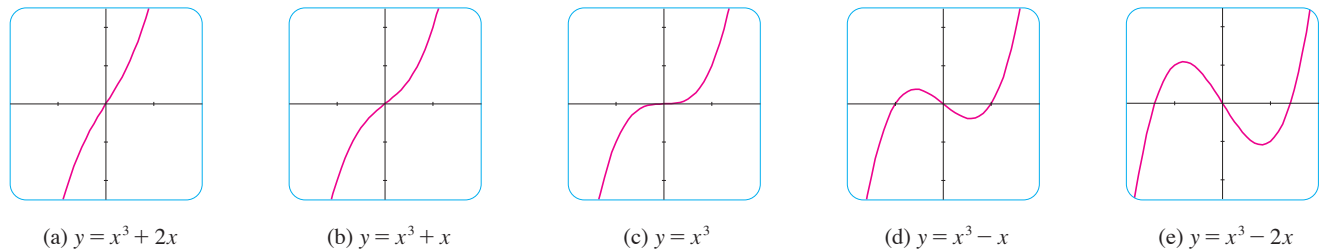


FIGURE 13

Several members of the family of functions $y = x^3 + cx$, all graphed in the viewing rectangle $[-2, 2]$ by $[-2.5, 2.5]$

EXAMPLE 9 Find the solution of the equation $\cos x = x$ correct to two decimal places.

SOLUTION The solutions of the equation $\cos x = x$ are the x -coordinates of the points of intersection of the curves $y = \cos x$ and $y = x$. From Figure 14(a) we see that there is only one solution and it lies between 0 and 1. Zooming in to the viewing rectangle $[0, 1]$ by $[0, 1]$, we see from Figure 14(b) that the root lies between 0.7 and 0.8. So we zoom in further to the viewing rectangle $[0.7, 0.8]$ by $[0.7, 0.8]$ in Figure 14(c). By moving the cursor to the intersection point of the two curves, or by inspection and the fact that the x -scale is 0.01, we see that the solution of the equation is about 0.74. (Many calculators have a built-in intersection feature.)

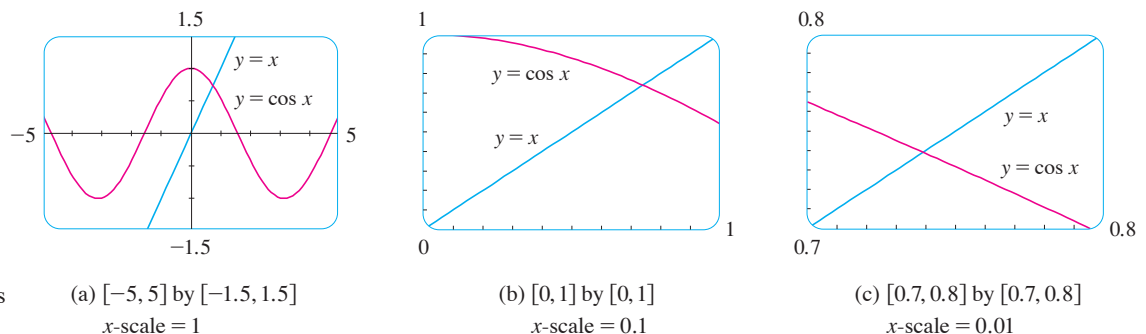


FIGURE 14

Locating the roots of $\cos x = x$

1.4 Exercises

- Use a graphing calculator or computer to determine which of the given viewing rectangles produces the most appropriate graph of the function $f(x) = \sqrt{x^3 - 5x^2}$.
(a) $[-5, 5]$ by $[-5, 5]$ (b) $[0, 10]$ by $[0, 2]$
(c) $[0, 10]$ by $[0, 10]$
 - Use a graphing calculator or computer to determine which of the given viewing rectangles produces the most appropriate graph of the function $f(x) = x^4 - 16x^2 + 20$.
(a) $[-3, 3]$ by $[-3, 3]$ (b) $[-10, 10]$ by $[-10, 10]$
(c) $[-50, 50]$ by $[-50, 50]$ (d) $[-5, 5]$ by $[-50, 50]$
- 3–14** Determine an appropriate viewing rectangle for the given function and use it to draw the graph.
- $f(x) = x^2 - 36x + 32$
 - $f(x) = x^3 + 15x^2 + 65x$
 - $f(x) = \sqrt{50 - 0.2x}$
 - $f(x) = \sqrt{15x - x^2}$
 - $f(x) = x^3 - 225x$
 - $f(x) = \frac{x}{x^2 + 100}$
 - $f(x) = \sin^2(1000x)$
 - $f(x) = \cos(0.001x)$
 - $f(x) = \sin \sqrt{x}$
 - $f(x) = \sec(20\pi x)$
 - $y = 10 \sin x + \sin 100x$
 - $y = x^2 + 0.02 \sin 50x$
-
- (a) Try to find an appropriate viewing rectangle for $f(x) = (x - 10)^3 2^{-x}$.
(b) Do you need more than one window? Why?
 - Graph the function $f(x) = x^2 \sqrt{30 - x}$ in an appropriate viewing rectangle. Why does part of the graph appear to be missing?
 - Graph the ellipse $4x^2 + 2y^2 = 1$ by graphing the functions whose graphs are the upper and lower halves of the ellipse.
 - Graph the hyperbola $y^2 - 9x^2 = 1$ by graphing the functions whose graphs are the upper and lower branches of the hyperbola.
- 19–20** Do the graphs intersect in the given viewing rectangle? If they do, how many points of intersection are there?
- $y = 3x^2 - 6x + 1$, $y = 0.23x - 2.25$; $[-1, 3]$ by $[-2.5, 1.5]$
 - $y = 6 - 4x - x^2$, $y = 3x + 18$; $[-6, 2]$ by $[-5, 20]$
-
- 21–23** Find all solutions of the equation correct to two decimal places.
- $x^4 - x = 1$
 - $\sqrt{x} = x^3 - 1$
 - $\tan x = \sqrt{1 - x^2}$
-
- We saw in Example 9 that the equation $\cos x = x$ has exactly one solution.
(a) Use a graph to show that the equation $\cos x = 0.3x$ has three solutions and find their values correct to two decimal places.
(b) Find an approximate value of m such that the equation $\cos x = mx$ has exactly two solutions.
 - Use graphs to determine which of the functions $f(x) = 10x^2$ and $g(x) = x^3/10$ is eventually larger (that is, larger when x is very large).
 - Use graphs to determine which of the functions $f(x) = x^4 - 100x^3$ and $g(x) = x^3$ is eventually larger.
 - For what values of x is it true that $|\tan x - x| < 0.01$ and $-\pi/2 < x < \pi/2$?
 - Graph the polynomials $P(x) = 3x^5 - 5x^3 + 2x$ and $Q(x) = 3x^5$ on the same screen, first using the viewing rectangle $[-2, 2]$ by $[-2, 2]$ and then changing to $[-10, 10]$ by $[-10,000, 10,000]$. What do you observe from these graphs?
 - In this exercise we consider the family of root functions $f(x) = \sqrt[n]{x}$, where n is a positive integer.
(a) Graph the functions $y = \sqrt{x}$, $y = \sqrt[4]{x}$, and $y = \sqrt[5]{x}$ on the same screen using the viewing rectangle $[-1, 4]$ by $[-1, 3]$.
(b) Graph the functions $y = x$, $y = \sqrt[3]{x}$, and $y = \sqrt[4]{x}$ on the same screen using the viewing rectangle $[-3, 3]$ by $[-2, 2]$. (See Example 7.)
(c) Graph the functions $y = \sqrt{x}$, $y = \sqrt[3]{x}$, $y = \sqrt[4]{x}$, and $y = \sqrt[5]{x}$ on the same screen using the viewing rectangle $[-1, 3]$ by $[-1, 2]$.
(d) What conclusions can you make from these graphs?
 - In this exercise we consider the family of functions $f(x) = 1/x^n$, where n is a positive integer.
(a) Graph the functions $y = 1/x$ and $y = 1/x^3$ on the same screen using the viewing rectangle $[-3, 3]$ by $[-3, 3]$.
(b) Graph the functions $y = 1/x^2$ and $y = 1/x^4$ on the same screen using the same viewing rectangle as in part (a).
(c) Graph all of the functions in parts (a) and (b) on the same screen using the viewing rectangle $[-1, 3]$ by $[-1, 3]$.
(d) What conclusions can you make from these graphs?
 - Graph the function $f(x) = x^4 + cx^2 + x$ for several values of c . How does the graph change when c changes?
 - Graph the function $f(x) = \sqrt{1 + cx^2}$ for various values of c . Describe how changing the value of c affects the graph.

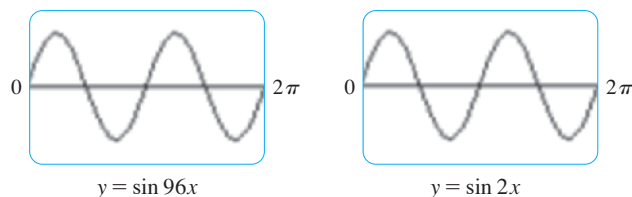
33. Graph the function $y = x^n 2^{-x}$, $x \geq 0$, for $n = 1, 2, 3, 4, 5$, and 6. How does the graph change as n increases?
34. The curves with equations

$$y = \frac{|x|}{\sqrt{c - x^2}}$$

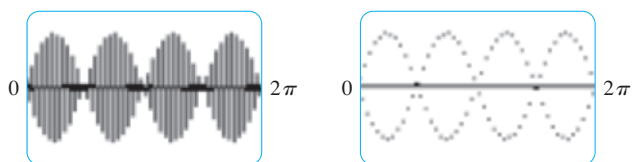
are called **bullet-nose curves**. Graph some of these curves to see why. What happens as c increases?

35. What happens to the graph of the equation $y^2 = cx^3 + x^2$ as c varies?
36. This exercise explores the effect of the inner function g on a composite function $y = f(g(x))$.
- Graph the function $y = \sin(\sqrt{x})$ using the viewing rectangle $[0, 400]$ by $[-1.5, 1.5]$. How does this graph differ from the graph of the sine function?
 - Graph the function $y = \sin(x^2)$ using the viewing rectangle $[-5, 5]$ by $[-1.5, 1.5]$. How does this graph differ from the graph of the sine function?
37. The figure shows the graphs of $y = \sin 96x$ and $y = \sin 2x$ as displayed by a TI-83 graphing calculator. The first graph is inaccurate. Explain why the two graphs appear identical.

[Hint: The TI-83's graphing window is 95 pixels wide. What specific points does the calculator plot?]



38. The first graph in the figure is that of $y = \sin 45x$ as displayed by a TI-83 graphing calculator. It is inaccurate and so, to help explain its appearance, we replot the curve in dot mode in the second graph. What two sine curves does the calculator appear to be plotting? Show that each point on the graph of $y = \sin 45x$ that the TI-83 chooses to plot is in fact on one of these two curves. (The TI-83's graphing window is 95 pixels wide.)



1.5 Exponential Functions

In Appendix G we present an alternative approach to the exponential and logarithmic functions using integral calculus.

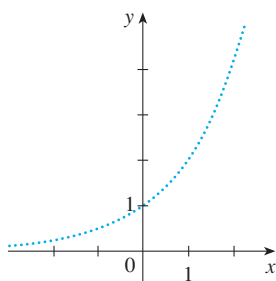


FIGURE 1
Representation of $y = 2^x$, x rational

The function $f(x) = 2^x$ is called an *exponential function* because the variable, x , is the exponent. It should not be confused with the power function $g(x) = x^2$, in which the variable is the base.

In general, an **exponential function** is a function of the form

$$f(x) = a^x$$

where a is a positive constant. Let's recall what this means.

If $x = n$, a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

If $x = 0$, then $a^0 = 1$, and if $x = -n$, where n is a positive integer, then

$$a^{-n} = \frac{1}{a^n}$$

If x is a rational number, $x = p/q$, where p and q are integers and $q > 0$, then

$$a^x = a^{p/q} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$$

But what is the meaning of a^x if x is an irrational number? For instance, what is meant by $2^{\sqrt{3}}$ or 5^π ?

To help us answer this question we first look at the graph of the function $y = 2^x$, where x is rational. A representation of this graph is shown in Figure 1. We want to enlarge the domain of $y = 2^x$ to include both rational and irrational numbers.

There are holes in the graph in Figure 1 corresponding to irrational values of x . We want to fill in the holes by defining $f(x) = 2^x$, where $x \in \mathbb{R}$, so that f is an increasing function. In particular, since the irrational number $\sqrt{3}$ satisfies

$$1.7 < \sqrt{3} < 1.8$$

we must have

$$2^{1.7} < 2^{\sqrt{3}} < 2^{1.8}$$

and we know what $2^{1.7}$ and $2^{1.8}$ mean because 1.7 and 1.8 are rational numbers. Similarly, if we use better approximations for $\sqrt{3}$, we obtain better approximations for $2^{\sqrt{3}}$:

$$\begin{array}{lll} 1.73 < \sqrt{3} < 1.74 & \Rightarrow & 2^{1.73} < 2^{\sqrt{3}} < 2^{1.74} \\ 1.732 < \sqrt{3} < 1.733 & \Rightarrow & 2^{1.732} < 2^{\sqrt{3}} < 2^{1.733} \\ 1.7320 < \sqrt{3} < 1.7321 & \Rightarrow & 2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321} \\ 1.73205 < \sqrt{3} < 1.73206 & \Rightarrow & 2^{1.73205} < 2^{\sqrt{3}} < 2^{1.73206} \\ \vdots & & \vdots \end{array}$$

It can be shown that there is exactly one number that is greater than all of the numbers

$$2^{1.7}, \quad 2^{1.73}, \quad 2^{1.732}, \quad 2^{1.7320}, \quad 2^{1.73205}, \quad \dots$$

and less than all of the numbers

$$2^{1.8}, \quad 2^{1.74}, \quad 2^{1.733}, \quad 2^{1.7321}, \quad 2^{1.73206}, \quad \dots$$

We define $2^{\sqrt{3}}$ to be this number. Using the preceding approximation process we can compute it correct to six decimal places:

$$2^{\sqrt{3}} \approx 3.321997$$

Similarly, we can define 2^x (or a^x , if $a > 0$) where x is any irrational number. Figure 2 shows how all the holes in Figure 1 have been filled to complete the graph of the function $f(x) = 2^x, x \in \mathbb{R}$.

The graphs of members of the family of functions $y = a^x$ are shown in Figure 3 for various values of the base a . Notice that all of these graphs pass through the same point $(0, 1)$ because $a^0 = 1$ for $a \neq 0$. Notice also that as the base a gets larger, the exponential function grows more rapidly (for $x > 0$).

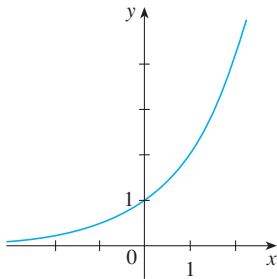


FIGURE 2
 $y = 2^x, x$ real

If $0 < a < 1$, then a^x approaches 0 as x becomes large. If $a > 1$, then a^x approaches 0 as x decreases through negative values. In both cases the x -axis is a horizontal asymptote. These matters are discussed in Section 2.6.

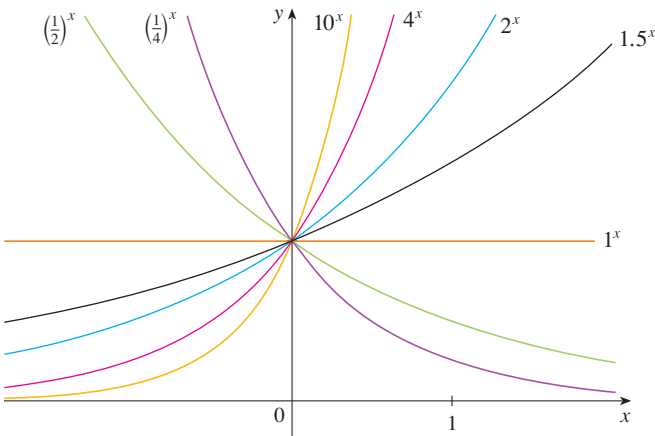


FIGURE 3

You can see from Figure 3 that there are basically three kinds of exponential functions $y = a^x$. If $0 < a < 1$, the exponential function decreases; if $a = 1$, it is a constant; and if $a > 1$, it increases. These three cases are illustrated in Figure 4. Observe that if $a \neq 1$, then the exponential function $y = a^x$ has domain \mathbb{R} and range $(0, \infty)$. Notice also that, since $(1/a)^x = 1/a^x = a^{-x}$, the graph of $y = (1/a)^x$ is just the reflection of the graph of $y = a^x$ about the y -axis.

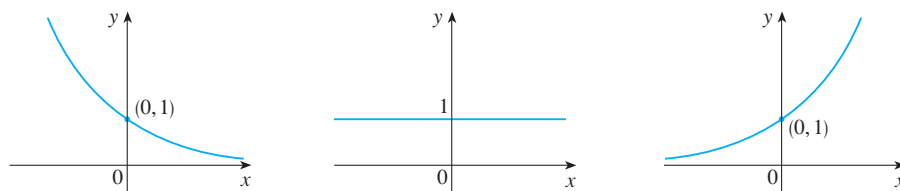


FIGURE 4

(a) $y = a^x$, $0 < a < 1$

(b) $y = 1^x$

(c) $y = a^x$, $a > 1$

One reason for the importance of the exponential function lies in the following properties. If x and y are rational numbers, then these laws are well known from elementary algebra. It can be proved that they remain true for arbitrary real numbers x and y .

www.stewartcalculus.com

For review and practice using the Laws of Exponents, click on *Review of Algebra*.

Laws of Exponents If a and b are positive numbers and x and y are any real numbers, then

$$1. a^{x+y} = a^x a^y \quad 2. a^{x-y} = \frac{a^x}{a^y} \quad 3. (a^x)^y = a^{xy} \quad 4. (ab)^x = a^x b^x$$

EXAMPLE 1 Sketch the graph of the function $y = 3 - 2^x$ and determine its domain and range.

SOLUTION First we reflect the graph of $y = 2^x$ [shown in Figures 2 and 5(a)] about the x -axis to get the graph of $y = -2^x$ in Figure 5(b). Then we shift the graph of $y = -2^x$ upward 3 units to obtain the graph of $y = 3 - 2^x$ in Figure 5(c). The domain is \mathbb{R} and the range is $(-\infty, 3)$.

For a review of reflecting and shifting graphs, see Section 1.3.

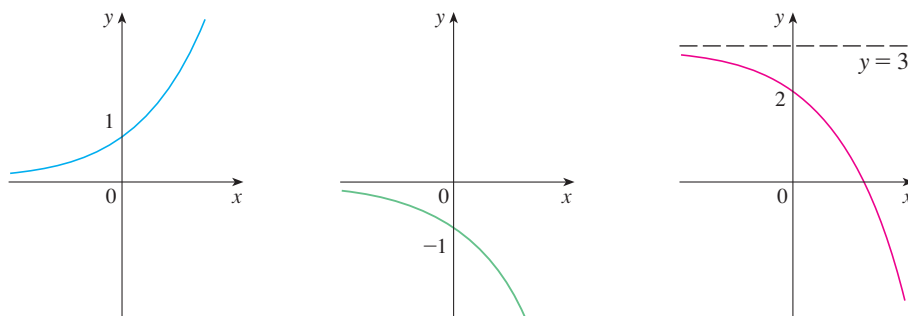


FIGURE 5

(a) $y = 2^x$

(b) $y = -2^x$

(c) $y = 3 - 2^x$

V EXAMPLE 2 Use a graphing device to compare the exponential function $f(x) = 2^x$ and the power function $g(x) = x^2$. Which function grows more quickly when x is large?

Example 2 shows that $y = 2^x$ increases more quickly than $y = x^2$. To demonstrate just how quickly $f(x) = 2^x$ increases, let's perform the following thought experiment. Suppose we start with a piece of paper a thousandth of an inch thick and we fold it in half 50 times. Each time we fold the paper in half, the thickness of the paper doubles, so the thickness of the resulting paper would be $2^{50}/1000$ inches. How thick do you think that is? It works out to be more than 17 million miles!

SOLUTION Figure 6 shows both functions graphed in the viewing rectangle $[-2, 6]$ by $[0, 40]$. We see that the graphs intersect three times, but for $x > 4$ the graph of $f(x) = 2^x$ stays above the graph of $g(x) = x^2$. Figure 7 gives a more global view and shows that for large values of x , the exponential function $y = 2^x$ grows far more rapidly than the power function $y = x^2$.

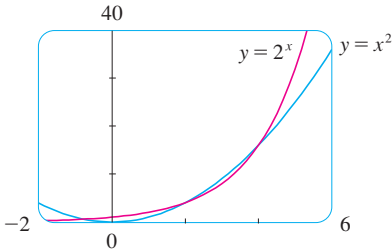


FIGURE 6

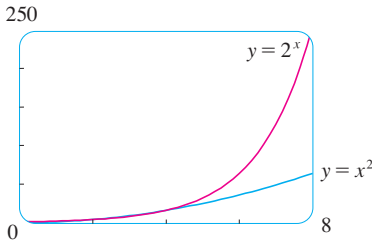


FIGURE 7

Applications of Exponential Functions

The exponential function occurs very frequently in mathematical models of nature and society. Here we indicate briefly how it arises in the description of population growth. In later chapters we will pursue these and other applications in greater detail.

First we consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour. If the number of bacteria at time t is $p(t)$, where t is measured in hours, and the initial population is $p(0) = 1000$, then we have

$$p(1) = 2p(0) = 2 \times 1000$$

$$p(2) = 2p(1) = 2^2 \times 1000$$

$$p(3) = 2p(2) = 2^3 \times 1000$$

It seems from this pattern that, in general,

$$p(t) = 2^t \times 1000 = (1000)2^t$$

This population function is a constant multiple of the exponential function $y = 2^t$, so it exhibits the rapid growth that we observed in Figures 2 and 7. Under ideal conditions (unlimited space and nutrition and absence of disease) this exponential growth is typical of what actually occurs in nature.

What about the human population? Table 1 shows data for the population of the world in the 20th century and Figure 8 shows the corresponding scatter plot.

TABLE 1

t	Population (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870

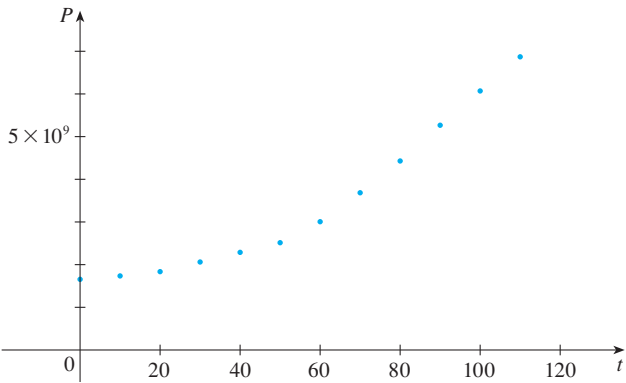


FIGURE 8 Scatter plot for world population growth

The pattern of the data points in Figure 8 suggests exponential growth, so we use a graphing calculator with exponential regression capability to apply the method of least squares and obtain the exponential model

$$P = (1436.53) \cdot (1.01395)^t$$

where $t = 0$ corresponds to 1900. Figure 9 shows the graph of this exponential function together with the original data points. We see that the exponential curve fits the data reasonably well. The period of relatively slow population growth is explained by the two world wars and the Great Depression of the 1930s.

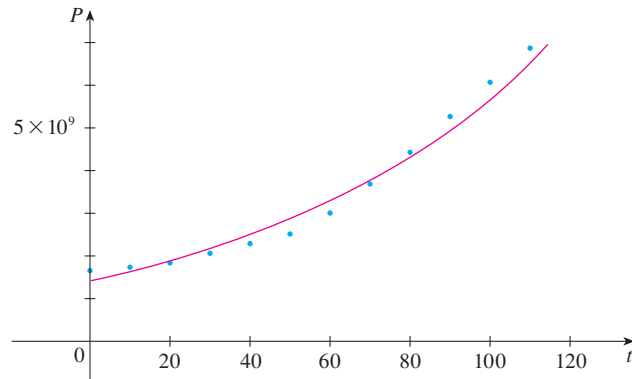


FIGURE 9
Exponential model for
population growth

The Number e

Of all possible bases for an exponential function, there is one that is most convenient for the purposes of calculus. The choice of a base a is influenced by the way the graph of $y = a^x$ crosses the y -axis. Figures 10 and 11 show the tangent lines to the graphs of $y = 2^x$ and $y = 3^x$ at the point $(0, 1)$. (Tangent lines will be defined precisely in Section 2.7. For present purposes, you can think of the tangent line to an exponential graph at a point as the line that touches the graph only at that point.) If we measure the slopes of these tangent lines at $(0, 1)$, we find that $m \approx 0.7$ for $y = 2^x$ and $m \approx 1.1$ for $y = 3^x$.

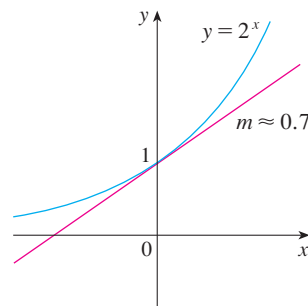


FIGURE 10

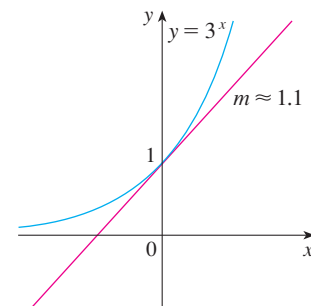


FIGURE 11

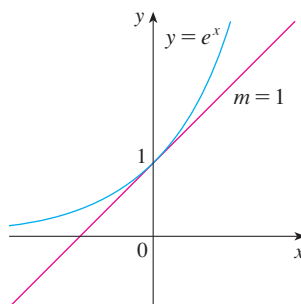


FIGURE 12
The natural exponential function
crosses the y -axis with a slope of 1.

It turns out, as we will see in Chapter 3, that some of the formulas of calculus will be greatly simplified if we choose the base a so that the slope of the tangent line to $y = a^x$ at $(0, 1)$ is *exactly* 1. (See Figure 12.) In fact, there *is* such a number and it is denoted by the letter e . (This notation was chosen by the Swiss mathematician Leonhard Euler in 1727, probably because it is the first letter of the word *exponential*.) In view of Figures 10 and 11, it comes as no surprise that the number e lies between 2 and 3 and the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$. (See Figure 13.) In Chapter 3 we will see that the value of e , correct to five decimal places, is

$$e \approx 2.71828$$

We call the function $f(x) = e^x$ the **natural exponential function**.

TEC Module 1.5 enables you to graph exponential functions with various bases and their tangent lines in order to estimate more closely the value of a for which the tangent has slope 1.

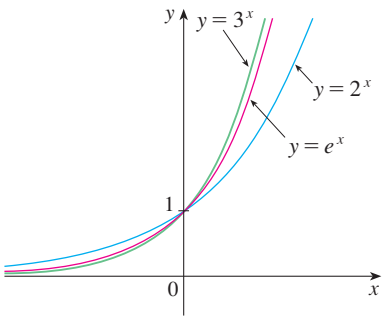


FIGURE 13

V EXAMPLE 3 Graph the function $y = \frac{1}{2}e^{-x} - 1$ and state the domain and range.

SOLUTION We start with the graph of $y = e^x$ from Figures 12 and 14(a) and reflect about the y -axis to get the graph of $y = e^{-x}$ in Figure 14(b). (Notice that the graph crosses the y -axis with a slope of -1 .) Then we compress the graph vertically by a factor of 2 to obtain the graph of $y = \frac{1}{2}e^{-x}$ in Figure 14(c). Finally, we shift the graph downward one unit to get the desired graph in Figure 14(d). The domain is \mathbb{R} and the range is $(-1, \infty)$.

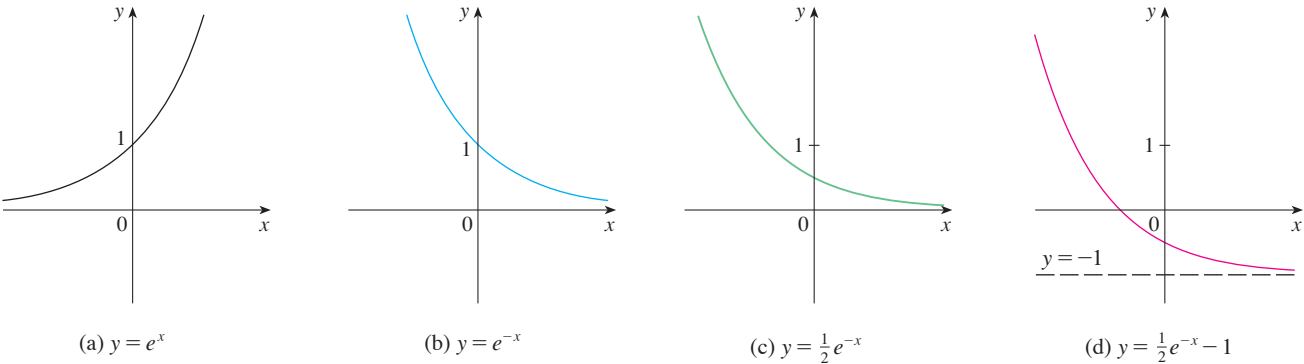


FIGURE 14

How far to the right do you think we would have to go for the height of the graph of $y = e^x$ to exceed a million? The next example demonstrates the rapid growth of this function by providing an answer that might surprise you.

EXAMPLE 4 Use a graphing device to find the values of x for which $e^x > 1,000,000$.

SOLUTION In Figure 15 we graph both the function $y = e^x$ and the horizontal line $y = 1,000,000$. We see that these curves intersect when $x \approx 13.8$. Thus $e^x > 10^6$ when $x > 13.8$. It is perhaps surprising that the values of the exponential function have already surpassed a million when x is only 14.

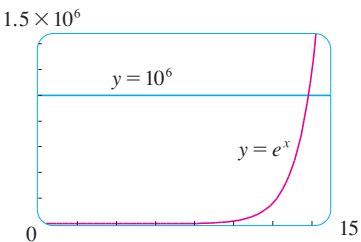


FIGURE 15

1.5 Exercises

1–4 Use the Law of Exponents to rewrite and simplify the expression.

1. (a) $\frac{4^{-3}}{2^{-8}}$ (b) $\frac{1}{\sqrt[3]{x^4}}$
2. (a) $8^{4/3}$ (b) $x(3x^2)^3$
3. (a) $b^8(2b)^4$ (b) $\frac{(6y^3)^4}{2y^5}$
4. (a) $\frac{x^{2n} \cdot x^{3n-1}}{x^{n+2}}$ (b) $\frac{\sqrt{a}\sqrt{b}}{\sqrt[3]{ab}}$

5. (a) Write an equation that defines the exponential function with base $a > 0$.
 (b) What is the domain of this function?
 (c) If $a \neq 1$, what is the range of this function?
 (d) Sketch the general shape of the graph of the exponential function for each of the following cases.
 (i) $a > 1$ (ii) $a = 1$ (iii) $0 < a < 1$
6. (a) How is the number e defined?
 (b) What is an approximate value for e ?
 (c) What is the natural exponential function?

 **7–10** Graph the given functions on a common screen. How are these graphs related?

7. $y = 2^x$, $y = e^x$, $y = 5^x$, $y = 20^x$
8. $y = e^x$, $y = e^{-x}$, $y = 8^x$, $y = 8^{-x}$
9. $y = 3^x$, $y = 10^x$, $y = \left(\frac{1}{3}\right)^x$, $y = \left(\frac{1}{10}\right)^x$
10. $y = 0.9^x$, $y = 0.6^x$, $y = 0.3^x$, $y = 0.1^x$

11–16 Make a rough sketch of the graph of the function. Do not use a calculator. Just use the graphs given in Figures 3 and 13 and, if necessary, the transformations of Section 1.3.

11. $y = 10^{x+2}$ 12. $y = (0.5)^x - 2$
13. $y = -2^{-x}$ 14. $y = e^{|x|}$
15. $y = 1 - \frac{1}{2}e^{-x}$ 16. $y = 2(1 - e^x)$

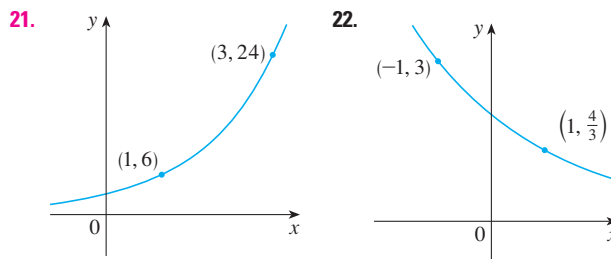
17. Starting with the graph of $y = e^x$, write the equation of the graph that results from
 (a) shifting 2 units downward
 (b) shifting 2 units to the right
 (c) reflecting about the x -axis
 (d) reflecting about the y -axis
 (e) reflecting about the x -axis and then about the y -axis

18. Starting with the graph of $y = e^x$, find the equation of the graph that results from
 (a) reflecting about the line $y = 4$
 (b) reflecting about the line $x = 2$

19–20 Find the domain of each function.



19. (a) $f(x) = \frac{1 - e^{x^2}}{1 - e^{1-x^2}}$ (b) $f(x) = \frac{1+x}{e^{\cos x}}$
20. (a) $g(t) = \sin(e^{-t})$ (b) $g(t) = \sqrt{1 - 2^t}$




21–22 Find the exponential function $f(x) = Ca^x$ whose graph is given.




23. If $f(x) = 5^x$, show that

$$\frac{f(x+h) - f(x)}{h} = 5^x \left(\frac{5^h - 1}{h} \right)$$

24. Suppose you are offered a job that lasts one month. Which of the following methods of payment do you prefer?
 I. One million dollars at the end of the month.
 II. One cent on the first day of the month, two cents on the second day, four cents on the third day, and, in general, 2^{n-1} cents on the n th day.
25. Suppose the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are drawn on a coordinate grid where the unit of measurement is 1 inch. Show that, at a distance 2 ft to the right of the origin, the height of the graph of f is 48 ft but the height of the graph of g is about 265 mi.
-  **26.** Compare the functions $f(x) = x^5$ and $g(x) = 5^x$ by graphing both functions in several viewing rectangles. Find all points of intersection of the graphs correct to one decimal place. Which function grows more rapidly when x is large?
-  **27.** Compare the functions $f(x) = x^{10}$ and $g(x) = e^x$ by graphing both f and g in several viewing rectangles. When does the graph of g finally surpass the graph of f ?

-  28. Use a graph to estimate the values of x such that $e^x > 1,000,000,000$.
29. Under ideal conditions a certain bacteria population is known to double every three hours. Suppose that there are initially 100 bacteria.
- (a) What is the size of the population after 15 hours?
 - (b) What is the size of the population after t hours?
 - (c) Estimate the size of the population after 20 hours.
 - (d) Graph the population function and estimate the time for the population to reach 50,000.
-  30. A bacterial culture starts with 500 bacteria and doubles in size every half hour.
- (a) How many bacteria are there after 3 hours?
 - (b) How many bacteria are there after t hours?
 - (c) How many bacteria are there after 40 minutes?
 - (d) Graph the population function and estimate the time for the population to reach 100,000.
-  31. Use a graphing calculator with exponential regression capability to model the population of the world with the data from 1950 to 2010 in Table 1 on page 54. Use the model to estimate the population in 1993 and to predict the population in the year 2020.

-  32. The table gives the population of the United States, in millions, for the years 1900–2010. Use a graphing calculator with exponential regression capability to model the US population since 1900. Use the model to estimate the population in 1925 and to predict the population in the year 2020.

Year	Population	Year	Population
1900	76	1960	179
1910	92	1970	203
1920	106	1980	227
1930	123	1990	250
1940	131	2000	281
1950	150	2010	310

-  33. If you graph the function

$$f(x) = \frac{1 - e^{1/x}}{1 + e^{1/x}}$$

you'll see that f appears to be an odd function. Prove it.

-  34. Graph several members of the family of functions

$$f(x) = \frac{1}{1 + ae^{bx}}$$

where $a > 0$. How does the graph change when b changes? How does it change when a changes?

1.6 Inverse Functions and Logarithms

Table 1 gives data from an experiment in which a bacteria culture started with 100 bacteria in a limited nutrient medium; the size of the bacteria population was recorded at hourly intervals. The number of bacteria N is a function of the time t : $N = f(t)$.

Suppose, however, that the biologist changes her point of view and becomes interested in the time required for the population to reach various levels. In other words, she is thinking of t as a function of N . This function is called the *inverse function* of f , denoted by f^{-1} , and read “ f inverse.” Thus $t = f^{-1}(N)$ is the time required for the population level to reach N . The values of f^{-1} can be found by reading Table 1 from right to left or by consulting Table 2. For instance, $f^{-1}(550) = 6$ because $f(6) = 550$.

TABLE 1 N as a function of t

t (hours)	$N = f(t)$ = population at time t
0	100
1	168
2	259
3	358
4	445
5	509
6	550
7	573
8	586

TABLE 2 t as a function of N

N	$t = f^{-1}(N)$ = time to reach N bacteria
100	0
168	1
259	2
358	3
445	4
509	5
550	6
573	7
586	8

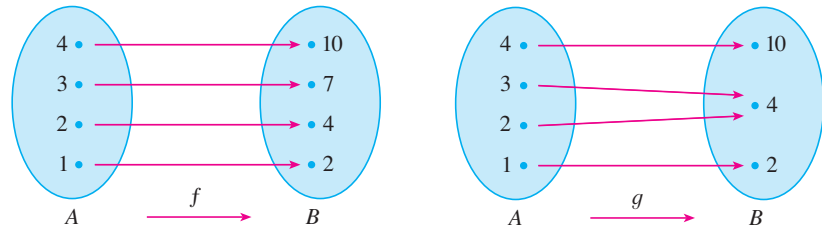
Not all functions possess inverses. Let's compare the functions f and g whose arrow diagrams are shown in Figure 1. Note that f never takes on the same value twice (any two inputs in A have different outputs), whereas g does take on the same value twice (both 2 and 3 have the same output, 4). In symbols,

$$g(2) = g(3)$$

but $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$

Functions that share this property with f are called *one-to-one functions*.

FIGURE 1
 f is one-to-one; g is not



In the language of inputs and outputs, this definition says that f is one-to-one if each output corresponds to only one input.

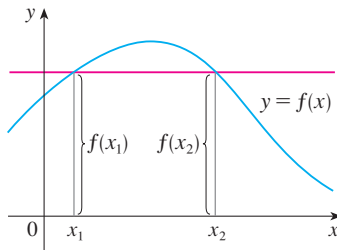


FIGURE 2
This function is not one-to-one because $f(x_1) = f(x_2)$.

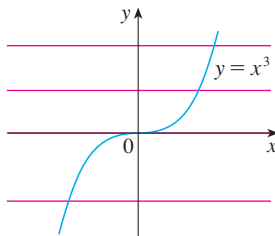


FIGURE 3
 $f(x) = x^3$ is one-to-one.

1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2$$

If a horizontal line intersects the graph of f in more than one point, then we see from Figure 2 that there are numbers x_1 and x_2 such that $f(x_1) = f(x_2)$. This means that f is not one-to-one. Therefore we have the following geometric method for determining whether a function is one-to-one.

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

V EXAMPLE 1 Is the function $f(x) = x^3$ one-to-one?

SOLUTION 1 If $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$ (two different numbers can't have the same cube). Therefore, by Definition 1, $f(x) = x^3$ is one-to-one.

SOLUTION 2 From Figure 3 we see that no horizontal line intersects the graph of $f(x) = x^3$ more than once. Therefore, by the Horizontal Line Test, f is one-to-one.

V EXAMPLE 2 Is the function $g(x) = x^2$ one-to-one?

SOLUTION 1 This function is not one-to-one because, for instance,

$$g(1) = 1 = g(-1)$$

and so 1 and -1 have the same output.

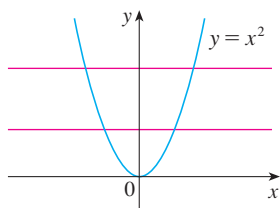


FIGURE 4
 $g(x) = x^2$ is not one-to-one.

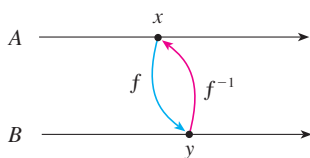


FIGURE 5

SOLUTION 2 From Figure 4 we see that there are horizontal lines that intersect the graph of g more than once. Therefore, by the Horizontal Line Test, g is not one-to-one.

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

2 Definition Let f be a one-to-one function with domain A and range B . Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B .

This definition says that if f maps x into y , then f^{-1} maps y back into x . (If f were not one-to-one, then f^{-1} would not be uniquely defined.) The arrow diagram in Figure 5 indicates that f^{-1} reverses the effect of f . Note that

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

For example, the inverse function of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$ because if $y = x^3$, then

$$f^{-1}(y) = f^{-1}(x^3) = (x^3)^{1/3} = x$$

CAUTION Do not mistake the -1 in f^{-1} for an exponent. Thus

$$f^{-1}(x) \text{ does not mean } \frac{1}{f(x)}$$

The reciprocal $1/f(x)$ could, however, be written as $[f(x)]^{-1}$.

V EXAMPLE 3 If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(7)$, $f^{-1}(5)$, and $f^{-1}(-10)$.

SOLUTION From the definition of f^{-1} we have

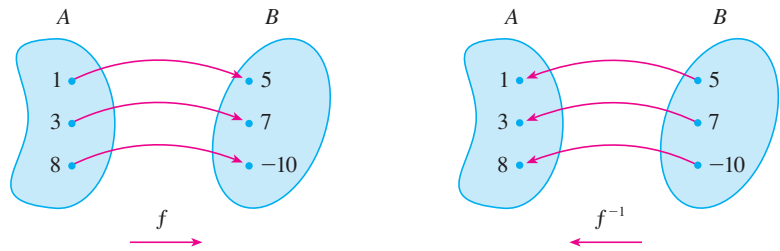
$$f^{-1}(7) = 3 \quad \text{because} \quad f(3) = 7$$

$$f^{-1}(5) = 1 \quad \text{because} \quad f(1) = 5$$

$$f^{-1}(-10) = 8 \quad \text{because} \quad f(8) = -10$$

The diagram in Figure 6 makes it clear how f^{-1} reverses the effect of f in this case.

FIGURE 6
The inverse function reverses
inputs and outputs.



The letter x is traditionally used as the independent variable, so when we concentrate on f^{-1} rather than on f , we usually reverse the roles of x and y in Definition 2 and write

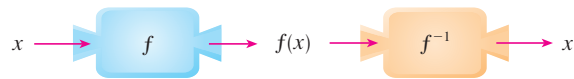
$$\boxed{3} \quad f^{-1}(x) = y \iff f(y) = x$$

By substituting for y in Definition 2 and substituting for x in $\boxed{3}$, we get the following **cancellation equations**:

$$\boxed{4} \quad \begin{aligned} f^{-1}(f(x)) &= x \quad \text{for every } x \text{ in } A \\ f(f^{-1}(x)) &= x \quad \text{for every } x \text{ in } B \end{aligned}$$

The first cancellation equation says that if we start with x , apply f , and then apply f^{-1} , we arrive back at x , where we started (see the machine diagram in Figure 7). Thus f^{-1} undoes what f does. The second equation says that f undoes what f^{-1} does.

FIGURE 7



For example, if $f(x) = x^3$, then $f^{-1}(x) = x^{1/3}$ and so the cancellation equations become

$$\begin{aligned} f^{-1}(f(x)) &= (x^3)^{1/3} = x \\ f(f^{-1}(x)) &= (x^{1/3})^3 = x \end{aligned}$$

These equations simply say that the cube function and the cube root function cancel each other when applied in succession.

Now let's see how to compute inverse functions. If we have a function $y = f(x)$ and are able to solve this equation for x in terms of y (if possible), then according to Definition 2 we must have $x = f^{-1}(y)$. If we want to call the independent variable x , we then interchange x and y and arrive at the equation $y = f^{-1}(x)$.

5 How to Find the Inverse Function of a One-to-One Function f

- Step 1 Write $y = f(x)$.
- Step 2 Solve this equation for x in terms of y (if possible).
- Step 3 To express f^{-1} as a function of x , interchange x and y .
The resulting equation is $y = f^{-1}(x)$.

V EXAMPLE 4 Find the inverse function of $f(x) = x^3 + 2$.

SOLUTION According to [5] we first write

$$y = x^3 + 2$$

Then we solve this equation for x :

$$x^3 = y - 2$$

$$x = \sqrt[3]{y - 2}$$

Finally, we interchange x and y :

$$y = \sqrt[3]{x - 2}$$

Therefore the inverse function is $f^{-1}(x) = \sqrt[3]{x - 2}$.

In Example 4, notice how f^{-1} reverses the effect of f . The function f is the rule “Cube, then add 2”; f^{-1} is the rule “Subtract 2, then take the cube root.”

The principle of interchanging x and y to find the inverse function also gives us the method for obtaining the graph of f^{-1} from the graph of f . Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1} . But we get the point (b, a) from (a, b) by reflecting about the line $y = x$. (See Figure 8.)

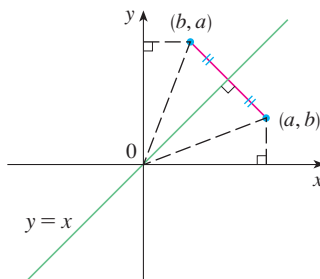


FIGURE 8

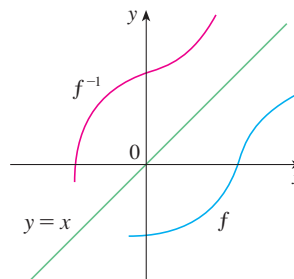


FIGURE 9

Therefore, as illustrated by Figure 9:

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

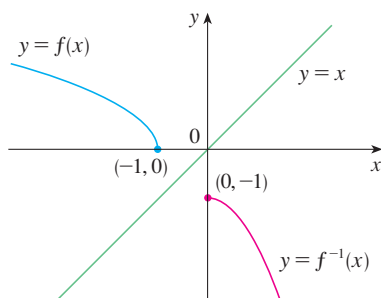


FIGURE 10

EXAMPLE 5 Sketch the graphs of $f(x) = \sqrt{-1 - x}$ and its inverse function using the same coordinate axes.

SOLUTION First we sketch the curve $y = \sqrt{-1 - x}$ (the top half of the parabola $y^2 = -1 - x$, or $x = -y^2 - 1$) and then we reflect about the line $y = x$ to get the graph of f^{-1} . (See Figure 10.) As a check on our graph, notice that the expression for f^{-1} is $f^{-1}(x) = -x^2 - 1, x \geq 0$. So the graph of f^{-1} is the right half of the parabola $y = -x^2 - 1$ and this seems reasonable from Figure 10.

Logarithmic Functions

If $a > 0$ and $a \neq 1$, the exponential function $f(x) = a^x$ is either increasing or decreasing and so it is one-to-one by the Horizontal Line Test. It therefore has an inverse function f^{-1} , which is called the **logarithmic function with base a** and is denoted by \log_a . If we use the

formulation of an inverse function given by [3],

$$f^{-1}(x) = y \iff f(y) = x$$

then we have

6

$$\log_a x = y \iff a^y = x$$

Thus, if $x > 0$, then $\log_a x$ is the exponent to which the base a must be raised to give x . For example, $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$.

The cancellation equations [4], when applied to the functions $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, become

7

$$\log_a(a^x) = x \quad \text{for every } x \in \mathbb{R}$$

$$a^{\log_a x} = x \quad \text{for every } x > 0$$

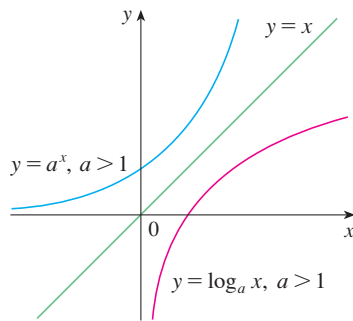


FIGURE 11

The logarithmic function \log_a has domain $(0, \infty)$ and range \mathbb{R} . Its graph is the reflection of the graph of $y = a^x$ about the line $y = x$.

Figure 11 shows the case where $a > 1$. (The most important logarithmic functions have base $a > 1$.) The fact that $y = a^x$ is a very rapidly increasing function for $x > 0$ is reflected in the fact that $y = \log_a x$ is a very slowly increasing function for $x > 1$.

Figure 12 shows the graphs of $y = \log_a x$ with various values of the base $a > 1$. Since $\log_a 1 = 0$, the graphs of all logarithmic functions pass through the point $(1, 0)$.

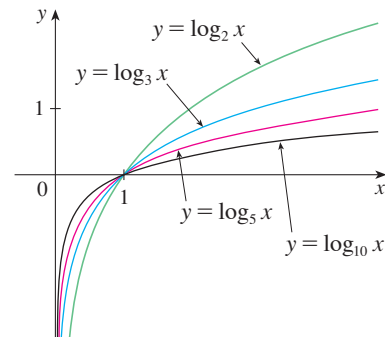


FIGURE 12

The following properties of logarithmic functions follow from the corresponding properties of exponential functions given in Section 1.5.

Laws of Logarithms If x and y are positive numbers, then

$$1. \log_a(xy) = \log_a x + \log_a y$$

$$2. \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$3. \log_a(x^r) = r \log_a x \quad (\text{where } r \text{ is any real number})$$

EXAMPLE 6 Use the laws of logarithms to evaluate $\log_2 80 - \log_2 5$.

SOLUTION Using Law 2, we have

$$\log_2 80 - \log_2 5 = \log_2 \left(\frac{80}{5} \right) = \log_2 16 = 4$$

because $2^4 = 16$.

Natural Logarithms

Notation for Logarithms

Most textbooks in calculus and the sciences, as well as calculators, use the notation $\ln x$ for the natural logarithm and $\log x$ for the “common logarithm,” $\log_{10} x$. In the more advanced mathematical and scientific literature and in computer languages, however, the notation $\log x$ usually denotes the natural logarithm.

$$\log_e x = \ln x$$

If we put $a = e$ and replace \log_e with “ \ln ” in [6] and [7], then the defining properties of the natural logarithm function become

8

$$\ln x = y \iff e^y = x$$

9

$$\ln(e^x) = x \quad x \in \mathbb{R}$$

$$e^{\ln x} = x \quad x > 0$$

In particular, if we set $x = 1$, we get

$$\ln e = 1$$

EXAMPLE 7 Find x if $\ln x = 5$.

SOLUTION 1 From [8] we see that

$$\ln x = 5 \quad \text{means} \quad e^5 = x$$

Therefore $x = e^5$.

(If you have trouble working with the “ \ln ” notation, just replace it by \log_e . Then the equation becomes $\log_e x = 5$; so, by the definition of logarithm, $e^5 = x$.)

SOLUTION 2 Start with the equation

$$\ln x = 5$$

and apply the exponential function to both sides of the equation:

$$e^{\ln x} = e^5$$

But the second cancellation equation in [9] says that $e^{\ln x} = x$. Therefore $x = e^5$.

EXAMPLE 8 Solve the equation $e^{5-3x} = 10$.

SOLUTION We take natural logarithms of both sides of the equation and use [9]:

$$\ln(e^{5-3x}) = \ln 10$$

$$5 - 3x = \ln 10$$

$$3x = 5 - \ln 10$$

$$x = \frac{1}{3}(5 - \ln 10)$$

Since the natural logarithm is found on scientific calculators, we can approximate the solution: to four decimal places, $x \approx 0.8991$.

V EXAMPLE 9 Express $\ln a + \frac{1}{2} \ln b$ as a single logarithm.

SOLUTION Using Laws 3 and 1 of logarithms, we have

$$\ln a + \frac{1}{2} \ln b = \ln a + \ln b^{1/2}$$

$$= \ln a + \ln \sqrt{b}$$

$$= \ln(a\sqrt{b})$$

The following formula shows that logarithms with any base can be expressed in terms of the natural logarithm.

[10] Change of Base Formula For any positive number a ($a \neq 1$), we have

$$\log_a x = \frac{\ln x}{\ln a}$$

PROOF Let $y = \log_a x$. Then, from [6], we have $a^y = x$. Taking natural logarithms of both sides of this equation, we get $y \ln a = \ln x$. Therefore

$$y = \frac{\ln x}{\ln a}$$

Scientific calculators have a key for natural logarithms, so Formula 10 enables us to use a calculator to compute a logarithm with any base (as shown in the following example). Similarly, Formula 10 allows us to graph any logarithmic function on a graphing calculator or computer (see Exercises 43 and 44).

EXAMPLE 10 Evaluate $\log_8 5$ correct to six decimal places.

SOLUTION Formula 10 gives

$$\log_8 5 = \frac{\ln 5}{\ln 8} \approx 0.773976$$

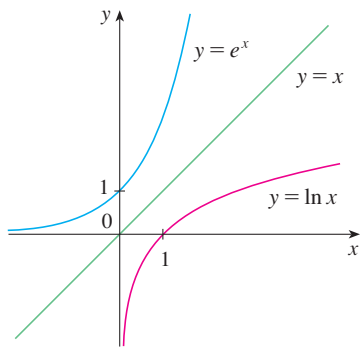


FIGURE 13
The graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ about the line $y = x$

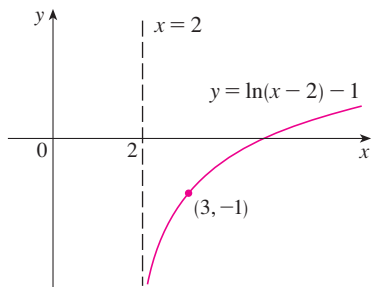
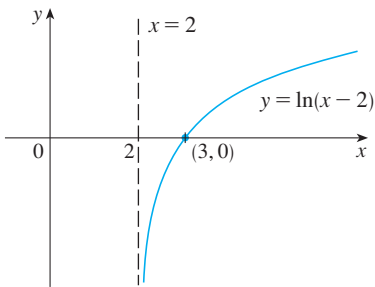
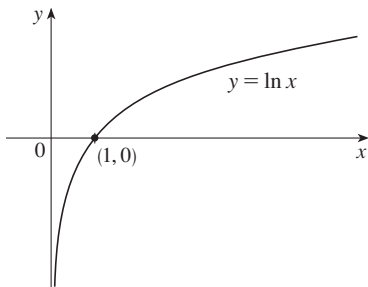


FIGURE 14

Graph and Growth of the Natural Logarithm

The graphs of the exponential function $y = e^x$ and its inverse function, the natural logarithm function, are shown in Figure 13. Because the curve $y = e^x$ crosses the y -axis with a slope of 1, it follows that the reflected curve $y = \ln x$ crosses the x -axis with a slope of 1.

In common with all other logarithmic functions with base greater than 1, the natural logarithm is an increasing function defined on $(0, \infty)$ and the y -axis is a vertical asymptote. (This means that the values of $\ln x$ become very large negative as x approaches 0.)

EXAMPLE 11 Sketch the graph of the function $y = \ln(x - 2) - 1$.

SOLUTION We start with the graph of $y = \ln x$ as given in Figure 13. Using the transformations of Section 1.3, we shift it 2 units to the right to get the graph of $y = \ln(x - 2)$ and then we shift it 1 unit downward to get the graph of $y = \ln(x - 2) - 1$. (See Figure 14.)

Although $\ln x$ is an increasing function, it grows *very* slowly when $x > 1$. In fact, $\ln x$ grows more slowly than any positive power of x . To illustrate this fact, we compare approximate values of the functions $y = \ln x$ and $y = x^{1/2} = \sqrt{x}$ in the following table and we graph them in Figures 15 and 16. You can see that initially the graphs of $y = \sqrt{x}$ and $y = \ln x$ grow at comparable rates, but eventually the root function far surpasses the logarithm.

x	1	2	5	10	50	100	500	1000	10,000	100,000
$\ln x$	0	0.69	1.61	2.30	3.91	4.6	6.2	6.9	9.2	11.5
\sqrt{x}	1	1.41	2.24	3.16	7.07	10.0	22.4	31.6	100	316
$\frac{\ln x}{\sqrt{x}}$	0	0.49	0.72	0.73	0.55	0.46	0.28	0.22	0.09	0.04

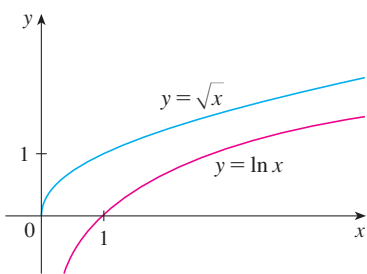


FIGURE 15

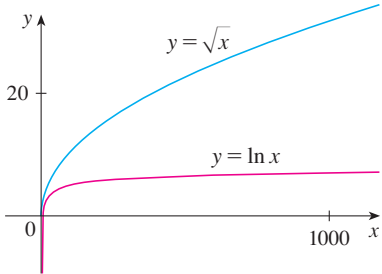


FIGURE 16

Inverse Trigonometric Functions

When we try to find the inverse trigonometric functions, we have a slight difficulty: Because the trigonometric functions are not one-to-one, they don't have inverse functions. The difficulty is overcome by restricting the domains of these functions so that they become one-to-one.

You can see from Figure 17 that the sine function $y = \sin x$ is not one-to-one (use the Horizontal Line Test). But the function $f(x) = \sin x$, $-\pi/2 \leq x \leq \pi/2$, is one-to-one (see Figure 18). The inverse function of this restricted sine function f exists and is denoted by \sin^{-1} or \arcsin . It is called the **inverse sine function** or the **arcsine function**.

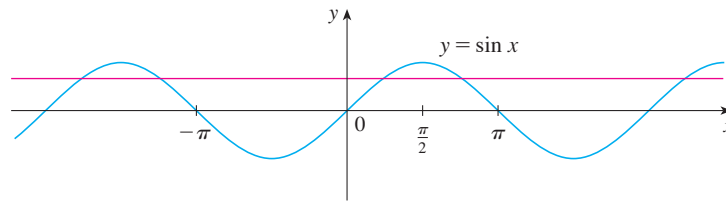
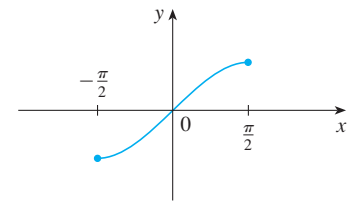


FIGURE 17

FIGURE 18 $y = \sin x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

Since the definition of an inverse function says that

$$f^{-1}(x) = y \iff f(y) = x$$

we have

$$\sin^{-1}x = y \iff \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

⊗ $\sin^{-1}x \neq \frac{1}{\sin x}$

Thus, if $-1 \leq x \leq 1$, $\sin^{-1}x$ is the number between $-\pi/2$ and $\pi/2$ whose sine is x .

EXAMPLE 12 Evaluate (a) $\sin^{-1}(\frac{1}{2})$ and (b) $\tan(\arcsin \frac{1}{3})$.

SOLUTION

(a) We have

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

because $\sin(\pi/6) = \frac{1}{2}$ and $\pi/6$ lies between $-\pi/2$ and $\pi/2$.

(b) Let $\theta = \arcsin \frac{1}{3}$, so $\sin \theta = \frac{1}{3}$. Then we can draw a right triangle with angle θ as in Figure 19 and deduce from the Pythagorean Theorem that the third side has length $\sqrt{9 - 1} = 2\sqrt{2}$. This enables us to read from the triangle that

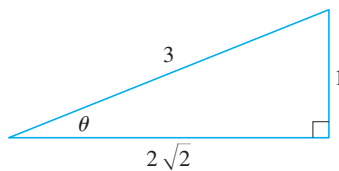


FIGURE 19

$$\tan(\arcsin \frac{1}{3}) = \tan \theta = \frac{1}{2\sqrt{2}}$$

The cancellation equations for inverse functions become, in this case,

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\sin(\sin^{-1}x) = x \quad \text{for } -1 \leq x \leq 1$$

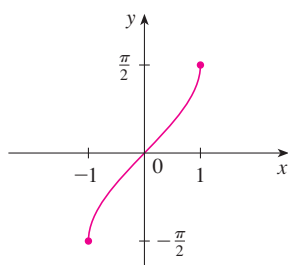


FIGURE 20

$$y = \sin^{-1} x = \arcsin x$$

The inverse sine function, \sin^{-1} , has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$, and its graph, shown in Figure 20, is obtained from that of the restricted sine function (Figure 18) by reflection about the line $y = x$.

The **inverse cosine function** is handled similarly. The restricted cosine function $f(x) = \cos x$, $0 \leq x \leq \pi$, is one-to-one (see Figure 21) and so it has an inverse function denoted by \cos^{-1} or arccos.

$$\cos^{-1} x = y \iff \cos y = x \quad \text{and} \quad 0 \leq y \leq \pi$$

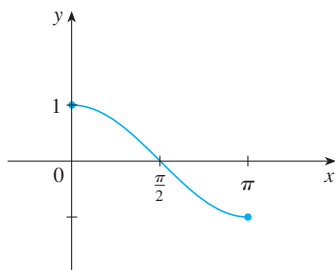


FIGURE 21

$$y = \cos x, 0 \leq x \leq \pi$$

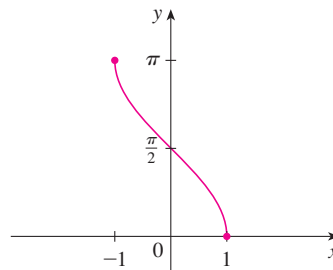


FIGURE 22

$$y = \cos^{-1} x = \arccos x$$

The cancellation equations are

$$\cos^{-1}(\cos x) = x \quad \text{for } 0 \leq x \leq \pi$$

$$\cos(\cos^{-1} x) = x \quad \text{for } -1 \leq x \leq 1$$

The inverse cosine function, \cos^{-1} , has domain $[-1, 1]$ and range $[0, \pi]$. Its graph is shown in Figure 22.

The tangent function can be made one-to-one by restricting it to the interval $(-\pi/2, \pi/2)$. Thus the **inverse tangent function** is defined as the inverse of the function $f(x) = \tan x$, $-\pi/2 < x < \pi/2$. (See Figure 23.) It is denoted by \tan^{-1} or arctan.

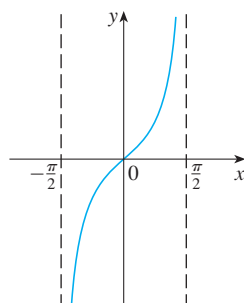


FIGURE 23

$$y = \tan x, -\pi/2 < x < \pi/2$$

$$\tan^{-1} x = y \iff \tan y = x \quad \text{and} \quad -\pi/2 < y < \pi/2$$

EXAMPLE 13 Simplify the expression $\cos(\tan^{-1} x)$.

SOLUTION 1 Let $y = \tan^{-1} x$. Then $\tan y = x$ and $-\pi/2 < y < \pi/2$. We want to find $\cos y$ but, since $\tan y$ is known, it is easier to find $\sec y$ first:

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$\sec y = \sqrt{1 + x^2} \quad (\text{since } \sec y > 0 \text{ for } -\pi/2 < y < \pi/2)$$

$$\text{Thus} \quad \cos(\tan^{-1} x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1 + x^2}}$$

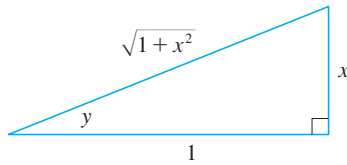


FIGURE 24

SOLUTION 2 Instead of using trigonometric identities as in Solution 1, it is perhaps easier to use a diagram. If $y = \tan^{-1}x$, then $\tan y = x$, and we can read from Figure 24 (which illustrates the case $y > 0$) that

$$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sqrt{1+x^2}}$$

The inverse tangent function, $\tan^{-1} = \arctan$, has domain \mathbb{R} and range $(-\pi/2, \pi/2)$. Its graph is shown in Figure 25.

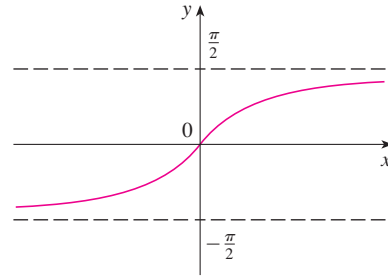


FIGURE 25
 $y = \tan^{-1}x = \arctan x$

We know that the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of \tan . Since the graph of \tan^{-1} is obtained by reflecting the graph of the restricted tangent function about the line $y = x$, it follows that the lines $y = \pi/2$ and $y = -\pi/2$ are horizontal asymptotes of the graph of \tan^{-1} .

The remaining inverse trigonometric functions are not used as frequently and are summarized here.

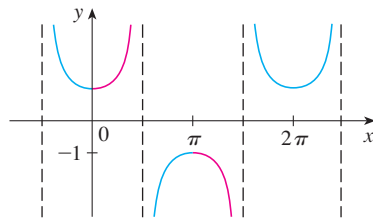


FIGURE 26
 $y = \sec x$

$$\boxed{11} \quad y = \csc^{-1}x \quad (|x| \geq 1) \iff \csc y = x \quad \text{and} \quad y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$

$$y = \sec^{-1}x \quad (|x| \geq 1) \iff \sec y = x \quad \text{and} \quad y \in [0, \pi/2) \cup [\pi, 3\pi/2)$$

$$y = \cot^{-1}x \quad (x \in \mathbb{R}) \iff \cot y = x \quad \text{and} \quad y \in (0, \pi)$$

The choice of intervals for y in the definitions of \csc^{-1} and \sec^{-1} is not universally agreed upon. For instance, some authors use $y \in [0, \pi/2) \cup (\pi/2, \pi]$ in the definition of \sec^{-1} . (You can see from the graph of the secant function in Figure 26 that both this choice and the one in $\boxed{11}$ will work.)

1.6 Exercises

- (a) What is a one-to-one function?
(b) How can you tell from the graph of a function whether it is one-to-one?
- (a) Suppose f is a one-to-one function with domain A and range B . How is the inverse function f^{-1} defined? What is the domain of f^{-1} ? What is the range of f^{-1} ?
(b) If you are given a formula for f , how do you find a formula for f^{-1} ?
(c) If you are given the graph of f , how do you find the graph of f^{-1} ?

3–14 A function is given by a table of values, a graph, a formula, or a verbal description. Determine whether it is one-to-one.

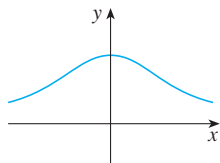
3.

x	1	2	3	4	5	6
$f(x)$	1.5	2.0	3.6	5.3	2.8	2.0

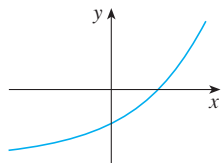
4.

x	1	2	3	4	5	6
$f(x)$	1.0	1.9	2.8	3.5	3.1	2.9

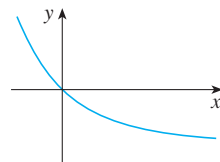
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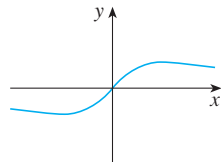
6.



7.



8.

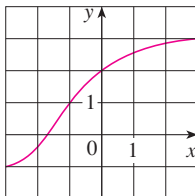


9. $f(x) = x^2 - 2x$

10. $f(x) = 10 - 3x$

11. $g(x) = 1/x$

12. $g(x) = \cos x$

13. $f(t)$ is the height of a football t seconds after kickoff.14. $f(t)$ is your height at age t .15. Assume that f is a one-to-one function.(a) If $f(6) = 17$, what is $f^{-1}(17)$?(b) If $f^{-1}(3) = 2$, what is $f(2)$?16. If $f(x) = x^5 + x^3 + x$, find $f^{-1}(3)$ and $f(f^{-1}(2))$.17. If $g(x) = 3 + x + e^x$, find $g^{-1}(4)$.18. The graph of f is given.(a) Why is f one-to-one?(b) What are the domain and range of f^{-1} ?(c) What is the value of $f^{-1}(2)$?(d) Estimate the value of $f^{-1}(0)$.

19. The formula $C = \frac{5}{9}(F - 32)$, where $F \geq -459.67$, expresses the Celsius temperature C as a function of the Fahrenheit temperature F . Find a formula for the inverse function and interpret it. What is the domain of the inverse function?

20. In the theory of relativity, the mass of a particle with speed v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and c is the speed of light in a vacuum. Find the inverse function of f and explain its meaning.

21–26 Find a formula for the inverse of the function.

21. $f(x) = 1 + \sqrt{2 + 3x}$

22. $f(x) = \frac{4x - 1}{2x + 3}$

23. $f(x) = e^{2x-1}$

24. $y = x^2 - x, \quad x \geq \frac{1}{2}$

25. $y = \ln(x + 3)$

26. $y = \frac{e^x}{1 + 2e^x}$

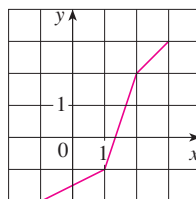
27–28 Find an explicit formula for f^{-1} and use it to graph f^{-1} , f , and the line $y = x$ on the same screen. To check your work, see whether the graphs of f and f^{-1} are reflections about the line.

27. $f(x) = x^4 + 1, \quad x \geq 0$

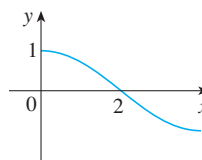
28. $f(x) = 2 - e^x$

29–30 Use the given graph of f to sketch the graph of f^{-1} .

29.



30.

31. Let $f(x) = \sqrt{1 - x^2}$, $0 \leq x \leq 1$.(a) Find f^{-1} . How is it related to f ?(b) Identify the graph of f and explain your answer to part (a).32. Let $g(x) = \sqrt[3]{1 - x^3}$.(a) Find g^{-1} . How is it related to g ?(b) Graph g . How do you explain your answer to part (a)?33. (a) How is the logarithmic function $y = \log_a x$ defined?

(b) What is the domain of this function?

(c) What is the range of this function?

(d) Sketch the general shape of the graph of the function $y = \log_a x$ if $a > 1$.

34. (a) What is the natural logarithm?

(b) What is the common logarithm?

(c) Sketch the graphs of the natural logarithm function and the natural exponential function with a common set of axes.

35–38 Find the exact value of each expression.

35. (a) $\log_5 125$

(b) $\log_3(\frac{1}{27})$

36. (a) $\ln(1/e)$

(b) $\log_{10} \sqrt{10}$

37. (a) $\log_2 6 - \log_2 15 + \log_2 20$

(b) $\log_3 100 - \log_3 18 - \log_3 50$

38. (a) $e^{-2 \ln 5}$

(b) $\ln(\ln e^{e^{10}})$

39–41 Express the given quantity as a single logarithm.

39. $\ln 5 + 5 \ln 3$


40. $\ln(a + b) + \ln(a - b) - 2 \ln c$

41. $\frac{1}{3} \ln(x + 2)^3 + \frac{1}{2} [\ln x - \ln(x^2 + 3x + 2)^2]$

42. Use Formula 10 to evaluate each logarithm correct to six decimal places.

(a) $\log_{12} 10$


(b) $\log_2 8.4$

 **43–44** Use Formula 10 to graph the given functions on a common screen. How are these graphs related?

43. $y = \log_{1.5} x$, $y = \ln x$, $y = \log_{10} x$, $y = \log_{50} x$

44. $y = \ln x$, $y = \log_{10} x$, $y = e^x$, $y = 10^x$

45. Suppose that the graph of $y = \log_2 x$ is drawn on a coordinate grid where the unit of measurement is an inch. How many miles to the right of the origin do we have to move before the height of the curve reaches 3 ft?

 **46.** Compare the functions $f(x) = x^{0.1}$ and $g(x) = \ln x$ by graphing both f and g in several viewing rectangles. When does the graph of f finally surpass the graph of g ?

47–48 Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graphs given in Figures 12 and 13 and, if necessary, the transformations of Section 1.3.

47. (a) $y = \log_{10}(x + 5)$ (b) $y = -\ln x$

48. (a) $y = \ln(-x)$ (b) $y = \ln |x|$

49–50 (a) What are the domain and range of f ?
(b) What is the x -intercept of the graph of f ?
(c) Sketch the graph of f .

49. $f(x) = \ln x + 2$

50. $f(x) = \ln(x - 1) - 1$

51–54 Solve each equation for x .

51. (a) $e^{7-4x} = 6$

(b) $\ln(3x - 10) = 2$

52. (a) $\ln(x^2 - 1) = 3$

(b) $e^{2x} - 3e^x + 2 = 0$

53. (a) $2^{x-5} = 3$

(b) $\ln x + \ln(x - 1) = 1$

54. (a) $\ln(\ln x) = 1$

(b) $e^{ax} = Ce^{bx}$, where $a \neq b$

55–56 Solve each inequality for x .

55. (a) $\ln x < 0$


(b) $e^x > 5$


56. (a) $1 < e^{3x-1} < 2$

(b) $1 - 2 \ln x < 3$

57. (a) Find the domain of $f(x) = \ln(e^x - 3)$.
(b) Find f^{-1} and its domain.

58. (a) What are the values of $e^{\ln 300}$ and $\ln(e^{300})$?
(b) Use your calculator to evaluate $e^{\ln 300}$ and $\ln(e^{300})$. What do you notice? Can you explain why the calculator has trouble?

 **59.** Graph the function $f(x) = \sqrt{x^3 + x^2 + x + 1}$ and explain why it is one-to-one. Then use a computer algebra system to find an explicit expression for $f^{-1}(x)$. (Your CAS will produce three possible expressions. Explain why two of them are irrelevant in this context.)

 **60.** (a) If $g(x) = x^6 + x^4$, $x \geq 0$, use a computer algebra system to find an expression for $g^{-1}(x)$.

(b) Use the expression in part (a) to graph $y = g(x)$, $y = x$, and $y = g^{-1}(x)$ on the same screen.

61. If a bacteria population starts with 100 bacteria and doubles every three hours, then the number of bacteria after t hours is $n = f(t) = 100 \cdot 2^{t/3}$. (See Exercise 29 in Section 1.5.)

(a) Find the inverse of this function and explain its meaning.

(b) When will the population reach 50,000?

62. When a camera flash goes off, the batteries immediately begin to recharge the flash's capacitor, which stores electric charge given by

$$Q(t) = Q_0(1 - e^{-t/a})$$

(The maximum charge capacity is Q_0 and t is measured in seconds.)

(a) Find the inverse of this function and explain its meaning.

(b) How long does it take to recharge the capacitor to 90% of capacity if $a = 2$?

63–68 Find the exact value of each expression.

63. (a) $\sin^{-1}(\sqrt{3}/2)$

(b) $\cos^{-1}(-1)$

64. (a) $\tan^{-1}(1/\sqrt{3})$

(b) $\sec^{-1} 2$

65. (a) $\arctan 1$

(b) $\sin^{-1}(1/\sqrt{2})$

66. (a) $\cot^{-1}(-\sqrt{3})$

(b) $\arccos(-\frac{1}{2})$

67. (a) $\tan(\arctan 10)$

(b) $\sin^{-1}(\sin(7\pi/3))$

68. (a) $\tan(\sec^{-1} 4)$

(b) $\sin(2 \sin^{-1}(\frac{3}{5}))$


69. Prove that $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$.

70–72 Simplify the expression.

70. $\tan(\sin^{-1} x)$

71. $\sin(\tan^{-1} x)$

72. $\cos(2 \tan^{-1} x)$


 **73–74** Graph the given functions on the same screen. How are these graphs related?

73. $y = \sin x$, $-\pi/2 \leq x \leq \pi/2$; $y = \sin^{-1} x$; $y = x$

74. $y = \tan x$, $-\pi/2 < x < \pi/2$; $y = \tan^{-1} x$; $y = x$

75. Find the domain and range of the function

$$g(x) = \sin^{-1}(3x + 1)$$

 **76.** (a) Graph the function $f(x) = \sin(\sin^{-1} x)$ and explain the appearance of the graph.

(b) Graph the function $g(x) = \sin^{-1}(\sin x)$. How do you explain the appearance of this graph?

77. (a) If we shift a curve to the left, what happens to its reflection about the line $y = x$? In view of this geometric principle, find an expression for the inverse of $g(x) = f(x + c)$, where f is a one-to-one function.

(b) Find an expression for the inverse of $h(x) = f(cx)$, where $c \neq 0$.

1 Review

Concept Check

- What is a function? What are its domain and range?
 - What is the graph of a function?
 - How can you tell whether a given curve is the graph of a function?
- Discuss four ways of representing a function. Illustrate your discussion with examples.
- What is an even function? How can you tell if a function is even by looking at its graph? Give three examples of an even function.
 - What is an odd function? How can you tell if a function is odd by looking at its graph? Give three examples of an odd function.
- What is an increasing function?
- What is a mathematical model?
- Give an example of each type of function.
 - Linear function
 - Power function
 - Exponential function
 - Quadratic function
 - Polynomial of degree 5
 - Rational function
- Sketch by hand, on the same axes, the graphs of the following functions.
 - $f(x) = x$
 - $g(x) = x^2$
 - $h(x) = x^3$
 - $j(x) = x^4$
- Draw, by hand, a rough sketch of the graph of each function.
 - $y = \sin x$
 - $y = \tan x$
 - $y = e^x$
 - $y = \ln x$
 - $y = 1/x$
 - $y = |x|$
 - $y = \sqrt{x}$
 - $y = \tan^{-1}x$
- Suppose that f has domain A and g has domain B .
 - What is the domain of $f + g$?
 - What is the domain of fg ?
 - What is the domain of f/g ?
- How is the composite function $f \circ g$ defined? What is its domain?
- Suppose the graph of f is given. Write an equation for each of the graphs that are obtained from the graph of f as follows.
 - Shift 2 units upward.
 - Shift 2 units downward.
 - Shift 2 units to the right.
 - Shift 2 units to the left.
 - Reflect about the x -axis.
 - Reflect about the y -axis.
 - Stretch vertically by a factor of 2.
 - Shrink vertically by a factor of 2.
 - Stretch horizontally by a factor of 2.
 - Shrink horizontally by a factor of 2.
- What is a one-to-one function? How can you tell if a function is one-to-one by looking at its graph?
 - If f is a one-to-one function, how is its inverse function f^{-1} defined? How do you obtain the graph of f^{-1} from the graph of f ?
- How is the inverse sine function $f(x) = \sin^{-1}x$ defined? What are its domain and range?
 - How is the inverse cosine function $f(x) = \cos^{-1}x$ defined? What are its domain and range?
 - How is the inverse tangent function $f(x) = \tan^{-1}x$ defined? What are its domain and range?

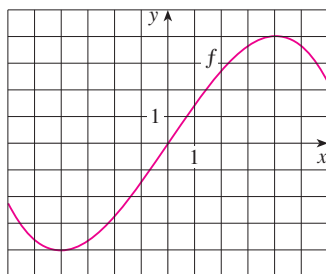
True-False Quiz

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

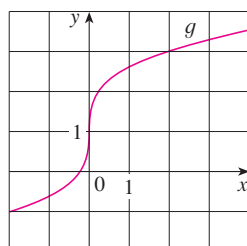
- If f is a function, then $f(s + t) = f(s) + f(t)$.
- If $f(s) = f(t)$, then $s = t$.
- If f is a function, then $f(3x) = 3f(x)$.
- If $x_1 < x_2$ and f is a decreasing function, then $f(x_1) > f(x_2)$.
- A vertical line intersects the graph of a function at most once.
- If f and g are functions, then $f \circ g = g \circ f$.
- If f is one-to-one, then $f^{-1}(x) = \frac{1}{f(x)}$.
- You can always divide by e^x .
- If $0 < a < b$, then $\ln a < \ln b$.
- If $x > 0$, then $(\ln x)^6 = 6 \ln x$.
- If $x > 0$ and $a > 1$, then $\frac{\ln x}{\ln a} = \ln \frac{x}{a}$.
- $\tan^{-1}(-1) = 3\pi/4$
- $\tan^{-1}x = \frac{\sin^{-1}x}{\cos^{-1}x}$
- If x is any real number, then $\sqrt{x^2} = x$.

Exercises

1. Let f be the function whose graph is given.
 - (a) Estimate the value of $f(2)$.
 - (b) Estimate the values of x such that $f(x) = 3$.
 - (c) State the domain of f .
 - (d) State the range of f .
 - (e) On what interval is f increasing?
 - (f) Is f one-to-one? Explain.
 - (g) Is f even, odd, or neither even nor odd? Explain.



2. The graph of g is given.
 - (a) State the value of $g(2)$.
 - (b) Why is g one-to-one?
 - (c) Estimate the value of $g^{-1}(2)$.
 - (d) Estimate the domain of g^{-1} .
 - (e) Sketch the graph of g^{-1} .



3. If $f(x) = x^2 - 2x + 3$, evaluate the difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

4. Sketch a rough graph of the yield of a crop as a function of the amount of fertilizer used.

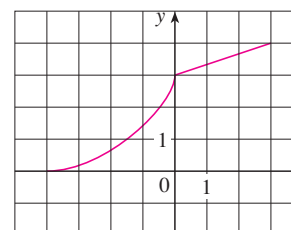
5–8 Find the domain and range of the function. Write your answer in interval notation.

5. $f(x) = 2/(3x - 1)$
6. $g(x) = \sqrt{16 - x^4}$
7. $h(x) = \ln(x + 6)$
8. $F(t) = 3 + \cos 2t$

9. Suppose that the graph of f is given. Describe how the graphs of the following functions can be obtained from the graph of f .
 - (a) $y = f(x) + 8$
 - (b) $y = f(x + 8)$
 - (c) $y = 1 + 2f(x)$
 - (d) $y = f(x - 2) - 2$
 - (e) $y = -f(x)$
 - (f) $y = f^{-1}(x)$

10. The graph of f is given. Draw the graphs of the following functions.

- (a) $y = f(x - 8)$
- (b) $y = -f(x)$
- (c) $y = 2 - f(x)$
- (d) $y = \frac{1}{2}f(x) - 1$
- (e) $y = f^{-1}(x)$
- (f) $y = f^{-1}(x + 3)$



11–16 Use transformations to sketch the graph of the function.

11. $y = -\sin 2x$
12. $y = 3 \ln(x - 2)$
13. $y = \frac{1}{2}(1 + e^x)$
14. $y = 2 - \sqrt{x}$

$$15. f(x) = \frac{1}{x + 2}$$

$$16. f(x) = \begin{cases} -x & \text{if } x < 0 \\ e^x - 1 & \text{if } x \geq 0 \end{cases}$$

17. Determine whether f is even, odd, or neither even nor odd.

- (a) $f(x) = 2x^5 - 3x^2 + 2$
- (b) $f(x) = x^3 - x^7$
- (c) $f(x) = e^{-x^2}$
- (d) $f(x) = 1 + \sin x$

18. Find an expression for the function whose graph consists of the line segment from the point $(-2, 2)$ to the point $(-1, 0)$ together with the top half of the circle with center the origin and radius 1.

19. If $f(x) = \ln x$ and $g(x) = x^2 - 9$, find the functions (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, (d) $g \circ g$, and their domains.

20. Express the function $F(x) = 1/\sqrt{x + \sqrt{x}}$ as a composition of three functions.

21. Life expectancy improved dramatically in the 20th century. The table gives the life expectancy at birth (in years) of males born in the United States. Use a scatter plot to choose an appropriate type of model. Use your model to predict the life span of a male born in the year 2010.

Birth year	Life expectancy	Birth year	Life expectancy
1900	48.3	1960	66.6
1910	51.1	1970	67.1
1920	55.2	1980	70.0
1930	57.4	1990	71.8
1940	62.5	2000	73.0
1950	65.6		

22. A small-appliance manufacturer finds that it costs \$9000 to produce 1000 toaster ovens a week and \$12,000 to produce 1500 toaster ovens a week.
- (a) Express the cost as a function of the number of toaster ovens produced, assuming that it is linear. Then sketch the graph.
- (b) What is the slope of the graph and what does it represent?
- (c) What is the y-intercept of the graph and what does it represent?
23. If $f(x) = 2x + \ln x$, find $f^{-1}(2)$.

24. Find the inverse function of $f(x) = \frac{x + 1}{2x + 1}$.
25. Find the exact value of each expression.
- (a) $e^{2 \ln 3}$ (b) $\log_{10} 25 + \log_{10} 4$
- (c) $\tan(\arcsin \frac{1}{2})$ (d) $\sin(\cos^{-1}(\frac{4}{5}))$
26. Solve each equation for x .
- (a) $e^x = 5$ (b) $\ln x = 2$
- (c) $e^{e^x} = 2$ (d) $\tan^{-1} x = 1$
27. The population of a certain species in a limited environment with initial population 100 and carrying capacity 1000 is

$$P(t) = \frac{100,000}{100 + 900e^{-t}}$$

where t is measured in years.



- (a) Graph this function and estimate how long it takes for the population to reach 900.
- (b) Find the inverse of this function and explain its meaning.
- (c) Use the inverse function to find the time required for the population to reach 900. Compare with the result of part (a).



28. Graph the three functions $y = x^a$, $y = a^x$, and $y = \log_a x$ on the same screen for two or three values of $a > 1$. For large values of x , which of these functions has the largest values and which has the smallest values?

Principles of Problem Solving

There are no hard and fast rules that will ensure success in solving problems. However, it is possible to outline some general steps in the problem-solving process and to give some principles that may be useful in the solution of certain problems. These steps and principles are just common sense made explicit. They have been adapted from George Polya's book *How To Solve It*.

1 UNDERSTAND THE PROBLEM

The first step is to read the problem and make sure that you understand it clearly. Ask yourself the following questions:

What is the unknown?

What are the given quantities?

What are the given conditions?

For many problems it is useful to

draw a diagram

and identify the given and required quantities on the diagram.

Usually it is necessary to

introduce suitable notation

In choosing symbols for the unknown quantities we often use letters such as a , b , c , m , n , x , and y , but in some cases it helps to use initials as suggestive symbols; for instance, V for volume or t for time.

2 THINK OF A PLAN

Find a connection between the given information and the unknown that will enable you to calculate the unknown. It often helps to ask yourself explicitly: "How can I relate the given to the unknown?" If you don't see a connection immediately, the following ideas may be helpful in devising a plan.

Try to Recognize Something Familiar Relate the given situation to previous knowledge. Look at the unknown and try to recall a more familiar problem that has a similar unknown.

Try to Recognize Patterns Some problems are solved by recognizing that some kind of pattern is occurring. The pattern could be geometric, or numerical, or algebraic. If you can see regularity or repetition in a problem, you might be able to guess what the continuing pattern is and then prove it.

Use Analogy Try to think of an analogous problem, that is, a similar problem, a related problem, but one that is easier than the original problem. If you can solve the similar, simpler problem, then it might give you the clues you need to solve the original, more difficult problem. For instance, if a problem involves very large numbers, you could first try a similar problem with smaller numbers. Or if the problem involves three-dimensional geometry, you could look for a similar problem in two-dimensional geometry. Or if the problem you start with is a general one, you could first try a special case.

Introduce Something Extra It may sometimes be necessary to introduce something new, an auxiliary aid, to help make the connection between the given and the unknown. For instance, in a problem where a diagram is useful the auxiliary aid could be a new line drawn in a diagram. In a more algebraic problem it could be a new unknown that is related to the original unknown.

Take Cases We may sometimes have to split a problem into several cases and give a different argument for each of the cases. For instance, we often have to use this strategy in dealing with absolute value.

Work Backward Sometimes it is useful to imagine that your problem is solved and work backward, step by step, until you arrive at the given data. Then you may be able to reverse your steps and thereby construct a solution to the original problem. This procedure is commonly used in solving equations. For instance, in solving the equation $3x - 5 = 7$, we suppose that x is a number that satisfies $3x - 5 = 7$ and work backward. We add 5 to each side of the equation and then divide each side by 3 to get $x = 4$. Since each of these steps can be reversed, we have solved the problem.

Establish Subgoals In a complex problem it is often useful to set subgoals (in which the desired situation is only partially fulfilled). If we can first reach these subgoals, then we may be able to build on them to reach our final goal.

Indirect Reasoning Sometimes it is appropriate to attack a problem indirectly. In using proof by contradiction to prove that P implies Q , we assume that P is true and Q is false and try to see why this can't happen. Somehow we have to use this information and arrive at a contradiction to what we absolutely know is true.

Mathematical Induction In proving statements that involve a positive integer n , it is frequently helpful to use the following principle.

Principle of Mathematical Induction Let S_n be a statement about the positive integer n . Suppose that

1. S_1 is true.
2. S_{k+1} is true whenever S_k is true.

Then S_n is true for all positive integers n .

This is reasonable because, since S_1 is true, it follows from condition 2 (with $k = 1$) that S_2 is true. Then, using condition 2 with $k = 2$, we see that S_3 is true. Again using condition 2, this time with $k = 3$, we have that S_4 is true. This procedure can be followed indefinitely.

3 CARRY OUT THE PLAN

In Step 2 a plan was devised. In carrying out that plan we have to check each stage of the plan and write the details that prove that each stage is correct.

4 LOOK BACK

Having completed our solution, it is wise to look back over it, partly to see if we have made errors in the solution and partly to see if we can think of an easier way to solve the problem. Another reason for looking back is that it will familiarize us with the method of solution and this may be useful for solving a future problem. Descartes said, "Every problem that I solved became a rule which served afterwards to solve other problems."

These principles of problem solving are illustrated in the following examples. Before you look at the solutions, try to solve these problems yourself, referring to these Principles of Problem Solving if you get stuck. You may find it useful to refer to this section from time to time as you solve the exercises in the remaining chapters of this book.

PS Understand the problem

PS Draw a diagram

PS Connect the given with the unknown

PS Introduce something extra

PS Relate to the familiar

EXAMPLE 1 Express the hypotenuse h of a right triangle with area 25 m^2 as a function of its perimeter P .

SOLUTION Let's first sort out the information by identifying the unknown quantity and the data:

Unknown: hypotenuse h

Given quantities: perimeter P , area 25 m^2

It helps to draw a diagram and we do so in Figure 1.

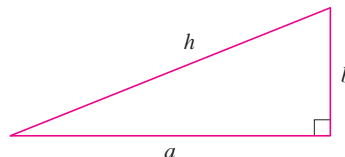


FIGURE 1

In order to connect the given quantities to the unknown, we introduce two extra variables a and b , which are the lengths of the other two sides of the triangle. This enables us to express the given condition, which is that the triangle is right-angled, by the Pythagorean Theorem:

$$h^2 = a^2 + b^2$$

The other connections among the variables come by writing expressions for the area and perimeter:

$$25 = \frac{1}{2}ab \quad P = a + b + h$$

Since P is given, notice that we now have three equations in the three unknowns a , b , and h :

1

$$h^2 = a^2 + b^2$$

2

$$25 = \frac{1}{2}ab$$

3

$$P = a + b + h$$

Although we have the correct number of equations, they are not easy to solve in a straightforward fashion. But if we use the problem-solving strategy of trying to recognize something familiar, then we can solve these equations by an easier method. Look at the right sides of Equations 1, 2, and 3. Do these expressions remind you of anything familiar? Notice that they contain the ingredients of a familiar formula:

$$(a + b)^2 = a^2 + 2ab + b^2$$

Using this idea, we express $(a + b)^2$ in two ways. From Equations 1 and 2 we have

$$(a + b)^2 = (a^2 + b^2) + 2ab = h^2 + 4(25)$$

From Equation 3 we have

$$(a + b)^2 = (P - h)^2 = P^2 - 2Ph + h^2$$

Thus

$$h^2 + 100 = P^2 - 2Ph + h^2$$

$$2Ph = P^2 - 100$$

$$h = \frac{P^2 - 100}{2P}$$

This is the required expression for h as a function of P .

As the next example illustrates, it is often necessary to use the problem-solving principle of *taking cases* when dealing with absolute values.

EXAMPLE 2 Solve the inequality $|x - 3| + |x + 2| < 11$.

SOLUTION Recall the definition of absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

It follows that

$$\begin{aligned} |x - 3| &= \begin{cases} x - 3 & \text{if } x - 3 \geq 0 \\ -(x - 3) & \text{if } x - 3 < 0 \end{cases} \\ &= \begin{cases} x - 3 & \text{if } x \geq 3 \\ -x + 3 & \text{if } x < 3 \end{cases} \end{aligned}$$

Similarly

$$\begin{aligned} |x + 2| &= \begin{cases} x + 2 & \text{if } x + 2 \geq 0 \\ -(x + 2) & \text{if } x + 2 < 0 \end{cases} \\ &= \begin{cases} x + 2 & \text{if } x \geq -2 \\ -x - 2 & \text{if } x < -2 \end{cases} \end{aligned}$$

PS Take cases

These expressions show that we must consider three cases:

$$x < -2 \qquad -2 \leq x < 3 \qquad x \geq 3$$

CASE I If $x < -2$, we have

$$\begin{aligned} |x - 3| + |x + 2| &< 11 \\ -x + 3 - x - 2 &< 11 \\ -2x &< 10 \\ x &> -5 \end{aligned}$$

CASE II If $-2 \leq x < 3$, the given inequality becomes

$$\begin{aligned} -x + 3 + x + 2 &< 11 \\ 5 &< 11 \quad (\text{always true}) \end{aligned}$$

CASE III If $x \geq 3$, the inequality becomes

$$\begin{aligned} x - 3 + x + 2 &< 11 \\ 2x &< 12 \\ x &< 6 \end{aligned}$$

Combining cases I, II, and III, we see that the inequality is satisfied when $-5 < x < 6$. So the solution is the interval $(-5, 6)$.

In the following example we first guess the answer by looking at special cases and recognizing a pattern. Then we prove our conjecture by mathematical induction.

In using the Principle of Mathematical Induction, we follow three steps:

Step 1 Prove that S_n is true when $n = 1$.

Step 2 Assume that S_n is true when $n = k$ and deduce that S_n is true when $n = k + 1$.

Step 3 Conclude that S_n is true for all n by the Principle of Mathematical Induction.

EXAMPLE 3 If $f_0(x) = x/(x + 1)$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \dots$, find a formula for $f_n(x)$.

PS Analogy: Try a similar, simpler problem

SOLUTION We start by finding formulas for $f_n(x)$ for the special cases $n = 1, 2$, and 3 .

$$\begin{aligned} f_1(x) &= (f_0 \circ f_0)(x) = f_0(f_0(x)) = f_0\left(\frac{x}{x+1}\right) \\ &= \frac{\frac{x}{x+1}}{\frac{x}{x+1} + 1} = \frac{\frac{x}{x+1}}{\frac{2x+1}{x+1}} = \frac{x}{2x+1} \end{aligned}$$

$$\begin{aligned} f_2(x) &= (f_0 \circ f_1)(x) = f_0(f_1(x)) = f_0\left(\frac{x}{2x+1}\right) \\ &= \frac{\frac{x}{2x+1}}{\frac{x}{2x+1} + 1} = \frac{\frac{x}{2x+1}}{\frac{3x+1}{2x+1}} = \frac{x}{3x+1} \end{aligned}$$

$$\begin{aligned} f_3(x) &= (f_0 \circ f_2)(x) = f_0(f_2(x)) = f_0\left(\frac{x}{3x+1}\right) \\ &= \frac{\frac{x}{3x+1}}{\frac{x}{3x+1} + 1} = \frac{\frac{x}{3x+1}}{\frac{4x+1}{3x+1}} = \frac{x}{4x+1} \end{aligned}$$

PS Look for a pattern

We notice a pattern: The coefficient of x in the denominator of $f_n(x)$ is $n + 1$ in the three cases we have computed. So we make the guess that, in general,

$$\boxed{4} \quad f_n(x) = \frac{x}{(n+1)x+1}$$

To prove this, we use the Principle of Mathematical Induction. We have already verified that $\boxed{4}$ is true for $n = 1$. Assume that it is true for $n = k$, that is,

$$\begin{aligned}
 \text{Then } f_{k+1}(x) &= (f_0 \circ f_k)(x) = f_0(f_k(x)) = f_0\left(\frac{x}{(k+1)x+1}\right) \\
 &= \frac{\frac{x}{(k+1)x+1}}{\frac{x}{(k+1)x+1} + 1} = \frac{\frac{x}{(k+1)x+1}}{\frac{(k+2)x+1}{(k+1)x+1}} = \frac{x}{(k+2)x+1}
 \end{aligned}$$

This expression shows that [4] is true for $n = k + 1$. Therefore, by mathematical induction, it is true for all positive integers n . ■

Problems

1. One of the legs of a right triangle has length 4 cm. Express the length of the altitude perpendicular to the hypotenuse as a function of the length of the hypotenuse.
2. The altitude perpendicular to the hypotenuse of a right triangle is 12 cm. Express the length of the hypotenuse as a function of the perimeter.
3. Solve the equation $|2x - 1| - |x + 5| = 3$.
4. Solve the inequality $|x - 1| - |x - 3| \geq 5$.
5. Sketch the graph of the function $f(x) = |x^2 - 4|x| + 3|$.
6. Sketch the graph of the function $g(x) = |x^2 - 1| - |x^2 - 4|$.
7. Draw the graph of the equation $x + |x| = y + |y|$.
8. Sketch the region in the plane consisting of all points (x, y) such that

$$|x - y| + |x| - |y| \leq 2$$

9. The notation $\max\{a, b, \dots\}$ means the largest of the numbers a, b, \dots . Sketch the graph of each function.
 (a) $f(x) = \max\{x, 1/x\}$ (b) $f(x) = \max\{\sin x, \cos x\}$ (c) $f(x) = \max\{x^2, 2 + x, 2 - x\}$
10. Sketch the region in the plane defined by each of the following equations or inequalities.
 (a) $\max\{x, 2y\} = 1$ (b) $-1 \leq \max\{x, 2y\} \leq 1$ (c) $\max\{x, y^2\} = 1$
11. Evaluate $(\log_2 3)(\log_3 4)(\log_4 5) \cdots (\log_{31} 32)$.
12. (a) Show that the function $f(x) = \ln(x + \sqrt{x^2 + 1})$ is an odd function.
 (b) Find the inverse function of f .
13. Solve the inequality $\ln(x^2 - 2x - 2) \leq 0$.
14. Use indirect reasoning to prove that $\log_2 5$ is an irrational number.
15. A driver sets out on a journey. For the first half of the distance she drives at the leisurely pace of 30 mi/h; she drives the second half at 60 mi/h. What is her average speed on this trip?
16. Is it true that $f \circ (g + h) = f \circ g + f \circ h$?
17. Prove that if n is a positive integer, then $7^n - 1$ is divisible by 6.
18. Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.
19. If $f_0(x) = x^2$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \dots$, find a formula for $f_n(x)$.
20. (a) If $f_0(x) = \frac{1}{2-x}$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \dots$, find an expression for $f_n(x)$ and use mathematical induction to prove it.
 (b) Graph f_0, f_1, f_2, f_3 on the same screen and describe the effects of repeated composition.



Graphing calculator or computer required

2

Limits and Derivatives

A ball falls faster and faster as time passes. Galileo discovered that the distance fallen is proportional to the square of the time it has been falling. Calculus then enables us to calculate the speed of the ball at any time.



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In *A Preview of Calculus* (page 1) we saw how the idea of a limit underlies the various branches of calculus. It is therefore appropriate to begin our study of calculus by investigating limits and their properties. The special type of limit that is used to find tangents and velocities gives rise to the central idea in differential calculus, the derivative.

2.1 The Tangent and Velocity Problems

In this section we see how limits arise when we attempt to find the tangent to a curve or the velocity of an object.

The Tangent Problem

The word *tangent* is derived from the Latin word *tangens*, which means “touching.” Thus a tangent to a curve is a line that touches the curve. In other words, a tangent line should have the same direction as the curve at the point of contact. How can this idea be made precise?

For a circle we could simply follow Euclid and say that a tangent is a line that intersects the circle once and only once, as in Figure 1(a). For more complicated curves this definition is inadequate. Figure 1(b) shows two lines l and t passing through a point P on a curve C . The line l intersects C only once, but it certainly does not look like what we think of as a tangent. The line t , on the other hand, looks like a tangent but it intersects C twice.

To be specific, let's look at the problem of trying to find a tangent line t to the parabola $y = x^2$ in the following example.

V EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

SOLUTION We will be able to find an equation of the tangent line t as soon as we know its slope m . The difficulty is that we know only one point, P , on t , whereas we need two points to compute the slope. But observe that we can compute an approximation to m by choosing a nearby point $Q(x, x^2)$ on the parabola (as in Figure 2) and computing the slope m_{PQ} of the secant line PQ . [A **secant line**, from the Latin word *secans*, meaning cutting, is a line that cuts (intersects) a curve more than once.]

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5$$

The tables in the margin show the values of m_{PQ} for several values of x close to 1. The closer Q is to P , the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2. This suggests that the slope of the tangent line t should be $m = 2$.

We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

$$\lim_{Q \rightarrow P} m_{PQ} = m \quad \text{and} \quad \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$$

Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line (see Appendix B) to write the equation of the tangent line through $(1, 1)$ as

$$y - 1 = 2(x - 1) \quad \text{or} \quad y = 2x - 1$$

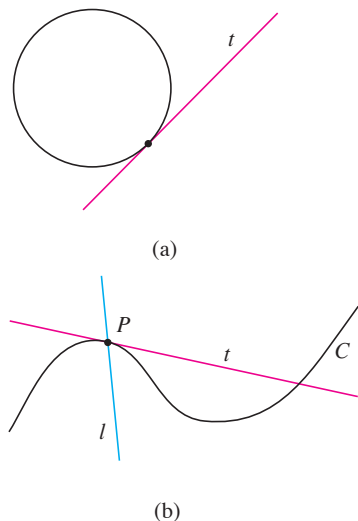


FIGURE 1

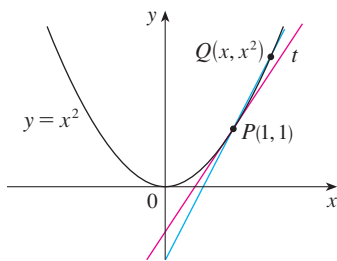


FIGURE 2

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

Figure 3 illustrates the limiting process that occurs in this example. As Q approaches P along the parabola, the corresponding secant lines rotate about P and approach the tangent line t .

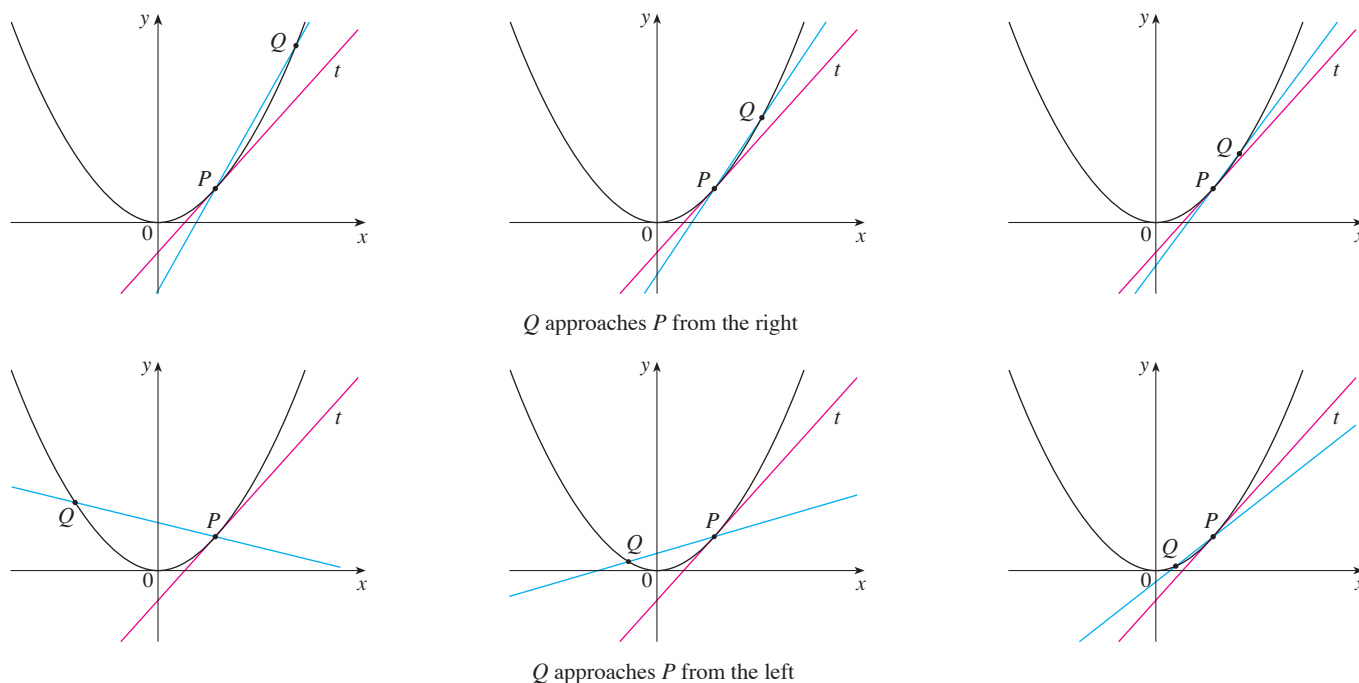


FIGURE 3

TEC In Visual 2.1 you can see how the process in Figure 3 works for additional functions.

t	Q
0.00	100.00
0.02	81.87
0.04	67.03
0.06	54.88
0.08	44.93
0.10	36.76

Many functions that occur in science are not described by explicit equations; they are defined by experimental data. The next example shows how to estimate the slope of the tangent line to the graph of such a function.

V EXAMPLE 2 The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The data in the table describe the charge Q remaining on the capacitor (measured in microcoulombs) at time t (measured in seconds after the flash goes off). Use the data to draw the graph of this function and estimate the slope of the tangent line at the point where $t = 0.04$. [Note: The slope of the tangent line represents the electric current flowing from the capacitor to the flash bulb (measured in microamperes).]

SOLUTION In Figure 4 we plot the given data and use them to sketch a curve that approximates the graph of the function.

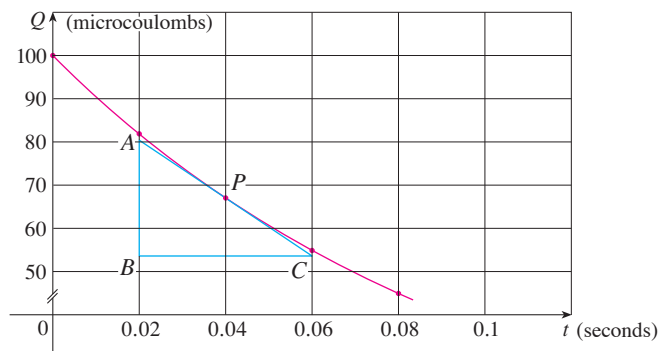


FIGURE 4

Given the points $P(0.04, 67.03)$ and $R(0.00, 100.00)$ on the graph, we find that the slope of the secant line PR is

$$m_{PR} = \frac{100.00 - 67.03}{0.00 - 0.04} = -824.25$$

R	m_{PR}
$(0.00, 100.00)$	-824.25
$(0.02, 81.87)$	-742.00
$(0.06, 54.88)$	-607.50
$(0.08, 44.93)$	-552.50
$(0.10, 36.76)$	-504.50

The table at the left shows the results of similar calculations for the slopes of other secant lines. From this table we would expect the slope of the tangent line at $t = 0.04$ to lie somewhere between -742 and -607.5 . In fact, the average of the slopes of the two closest secant lines is

$$\frac{1}{2}(-742 - 607.5) = -674.75$$

So, by this method, we estimate the slope of the tangent line to be -675 .

Another method is to draw an approximation to the tangent line at P and measure the sides of the triangle ABC , as in Figure 4. This gives an estimate of the slope of the tangent line as

$$-\frac{|AB|}{|BC|} \approx -\frac{80.4 - 53.6}{0.06 - 0.02} = -670$$

The physical meaning of the answer in Example 2 is that the electric current flowing from the capacitor to the flash bulb after 0.04 second is about -670 microamperes.

The Velocity Problem

If you watch the speedometer of a car as you travel in city traffic, you see that the needle doesn't stay still for very long; that is, the velocity of the car is not constant. We assume from watching the speedometer that the car has a definite velocity at each moment, but how is the "instantaneous" velocity defined? Let's investigate the example of a falling ball.

V EXAMPLE 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

SOLUTION Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.) If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation

$$s(t) = 4.9t^2$$

The difficulty in finding the velocity after 5 s is that we are dealing with a single instant of time ($t = 5$), so no time interval is involved. However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from $t = 5$ to $t = 5.1$:

$$\begin{aligned} \text{average velocity} &= \frac{\text{change in position}}{\text{time elapsed}} \\ &= \frac{s(5.1) - s(5)}{0.1} \\ &= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1} = 49.49 \text{ m/s} \end{aligned}$$



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The CN Tower in Toronto was the tallest free-standing building in the world for 32 years.

The following table shows the results of similar calculations of the average velocity over successively smaller time periods.

Time interval	Average velocity (m/s)
$5 \leq t \leq 6$	53.9
$5 \leq t \leq 5.1$	49.49
$5 \leq t \leq 5.05$	49.245
$5 \leq t \leq 5.01$	49.049
$5 \leq t \leq 5.001$	49.0049

It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The **instantaneous velocity** when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$. Thus the (instantaneous) velocity after 5 s is

$$v = 49 \text{ m/s}$$

You may have the feeling that the calculations used in solving this problem are very similar to those used earlier in this section to find tangents. In fact, there is a close connection between the tangent problem and the problem of finding velocities. If we draw the graph of the distance function of the ball (as in Figure 5) and we consider the points $P(a, 4.9a^2)$ and $Q(a + h, 4.9(a + h)^2)$ on the graph, then the slope of the secant line PQ is

$$m_{PQ} = \frac{4.9(a + h)^2 - 4.9a^2}{(a + h) - a}$$

which is the same as the average velocity over the time interval $[a, a + h]$. Therefore the velocity at time $t = a$ (the limit of these average velocities as h approaches 0) must be equal to the slope of the tangent line at P (the limit of the slopes of the secant lines).

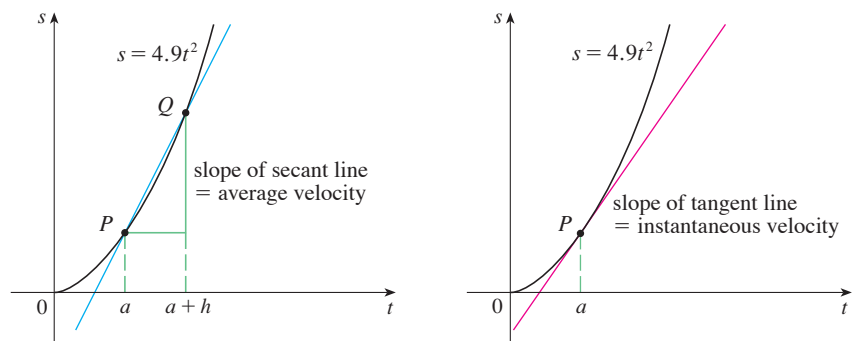


FIGURE 5

Examples 1 and 3 show that in order to solve tangent and velocity problems we must be able to find limits. After studying methods for computing limits in the next five sections, we will return to the problems of finding tangents and velocities in Section 2.7.

2.1 Exercises

1. A tank holds 1000 gallons of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume V of water remaining in the tank (in gallons) after t minutes.

t (min)	5	10	15	20	25	30
V (gal)	694	444	250	111	28	0

- (a) If P is the point $(15, 250)$ on the graph of V , find the slopes of the secant lines PQ when Q is the point on the graph with $t = 5, 10, 20, 25$, and 30 .
- (b) Estimate the slope of the tangent line at P by averaging the slopes of two secant lines.
- (c) Use a graph of the function to estimate the slope of the tangent line at P . (This slope represents the rate at which the water is flowing from the tank after 15 minutes.)
2. A cardiac monitor is used to measure the heart rate of a patient after surgery. It compiles the number of heartbeats after t minutes. When the data in the table are graphed, the slope of the tangent line represents the heart rate in beats per minute.

t (min)	36	38	40	42	44
Heartbeats	2530	2661	2806	2948	3080

The monitor estimates this value by calculating the slope of a secant line. Use the data to estimate the patient's heart rate after 42 minutes using the secant line between the points with the given values of t .

- (a) $t = 36$ and $t = 42$ (b) $t = 38$ and $t = 42$
 (c) $t = 40$ and $t = 42$ (d) $t = 42$ and $t = 44$

What are your conclusions?

3. The point $P(2, -1)$ lies on the curve $y = 1/(1 - x)$.
- (a) If Q is the point $(x, 1/(1 - x))$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x :
- (i) 1.5 (ii) 1.9 (iii) 1.99 (iv) 1.999
 (v) 2.5 (vi) 2.1 (vii) 2.01 (viii) 2.001
- (b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(2, -1)$.
- (c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(2, -1)$.
4. The point $P(0.5, 0)$ lies on the curve $y = \cos \pi x$.
- (a) If Q is the point $(x, \cos \pi x)$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x :
- (i) 0 (ii) 0.4 (iii) 0.49 (iv) 0.499
 (v) 1 (vi) 0.6 (vii) 0.51 (viii) 0.501
- (b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(0.5, 0)$.

- (c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(0.5, 0)$.
- (d) Sketch the curve, two of the secant lines, and the tangent line.

5. If a ball is thrown into the air with a velocity of 40 ft/s, its height in feet t seconds later is given by $y = 40t - 16t^2$.
- (a) Find the average velocity for the time period beginning when $t = 2$ and lasting
- (i) 0.5 second (ii) 0.1 second
 (iii) 0.05 second (iv) 0.01 second
- (b) Estimate the instantaneous velocity when $t = 2$.
6. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height in meters t seconds later is given by $y = 10t - 1.86t^2$.
- (a) Find the average velocity over the given time intervals:
- (i) $[1, 2]$ (ii) $[1, 1.5]$ (iii) $[1, 1.1]$
 (iv) $[1, 1.01]$ (v) $[1, 1.001]$
- (b) Estimate the instantaneous velocity when $t = 1$.
7. The table shows the position of a cyclist.

t (seconds)	0	1	2	3	4	5
s (meters)	0	1.4	5.1	10.7	17.7	25.8

- (a) Find the average velocity for each time period:
- (i) $[1, 3]$ (ii) $[2, 3]$ (iii) $[3, 5]$ (iv) $[3, 4]$
- (b) Use the graph of s as a function of t to estimate the instantaneous velocity when $t = 3$.
8. The displacement (in centimeters) of a particle moving back and forth along a straight line is given by the equation of motion $s = 2 \sin \pi t + 3 \cos \pi t$, where t is measured in seconds.
- (a) Find the average velocity during each time period:
- (i) $[1, 2]$ (ii) $[1, 1.1]$
 (iii) $[1, 1.01]$ (iv) $[1, 1.001]$
- (b) Estimate the instantaneous velocity of the particle when $t = 1$.
9. The point $P(1, 0)$ lies on the curve $y = \sin(10\pi/x)$.
- (a) If Q is the point $(x, \sin(10\pi/x))$, find the slope of the secant line PQ (correct to four decimal places) for $x = 2, 1.5, 1.4, 1.3, 1.2, 1.1, 0.5, 0.6, 0.7, 0.8$, and 0.9 . Do the slopes appear to be approaching a limit?
- (b) Use a graph of the curve to explain why the slopes of the secant lines in part (a) are not close to the slope of the tangent line at P .
- (c) By choosing appropriate secant lines, estimate the slope of the tangent line at P .

2.2 The Limit of a Function

Having seen in the preceding section how limits arise when we want to find the tangent to a curve or the velocity of an object, we now turn our attention to limits in general and numerical and graphical methods for computing them.

Let's investigate the behavior of the function f defined by $f(x) = x^2 - x + 2$ for values of x near 2. The following table gives values of $f(x)$ for values of x close to 2 but not equal to 2.

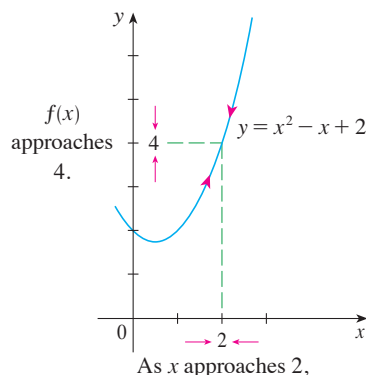


FIGURE 1

x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

From the table and the graph of f (a parabola) shown in Figure 1 we see that when x is close to 2 (on either side of 2), $f(x)$ is close to 4. In fact, it appears that we can make the values of $f(x)$ as close as we like to 4 by taking x sufficiently close to 2. We express this by saying “the limit of the function $f(x) = x^2 - x + 2$ as x approaches 2 is equal to 4.” The notation for this is

$$\lim_{x \rightarrow 2} (x^2 - x + 2) = 4$$

In general, we use the following notation.

1 Definition Suppose $f(x)$ is defined when x is near the number a . (This means that f is defined on some open interval that contains a , except possibly at a itself.) Then we write

$$\lim_{x \rightarrow a} f(x) = L$$

and say “the limit of $f(x)$, as x approaches a , equals L ”

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a .

Roughly speaking, this says that the values of $f(x)$ approach L as x approaches a . In other words, the values of $f(x)$ tend to get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$. (A more precise definition will be given in Section 2.4.)

An alternative notation for

$$\lim_{x \rightarrow a} f(x) = L$$

is $f(x) \rightarrow L$ as $x \rightarrow a$

which is usually read “ $f(x)$ approaches L as x approaches a .”

Notice the phrase “but $x \neq a$ ” in the definition of limit. This means that in finding the limit of $f(x)$ as x approaches a , we never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. The only thing that matters is how f is defined *near* a .

Figure 2 shows the graphs of three functions. Note that in part (c), $f(a)$ is not defined and in part (b), $f(a) \neq L$. But in each case, regardless of what happens at a , it is true that $\lim_{x \rightarrow a} f(x) = L$.

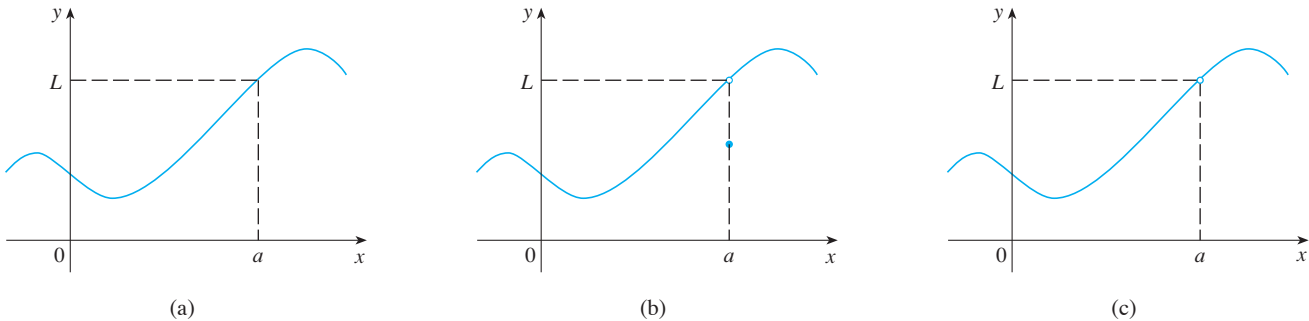


FIGURE 2 $\lim_{x \rightarrow a} f(x) = L$ in all three cases

EXAMPLE 1 Guess the value of $\lim_{x \rightarrow 1} \frac{x - 1}{x^2 - 1}$.

SOLUTION Notice that the function $f(x) = (x - 1)/(x^2 - 1)$ is not defined when $x = 1$, but that doesn’t matter because the definition of $\lim_{x \rightarrow a} f(x)$ says that we consider values of x that are close to a but not equal to a .

The tables at the left give values of $f(x)$ (correct to six decimal places) for values of x that approach 1 (but are not equal to 1). On the basis of the values in the tables, we make the guess that

$$\lim_{x \rightarrow 1} \frac{x - 1}{x^2 - 1} = 0.5$$

Example 1 is illustrated by the graph of f in Figure 3. Now let’s change f slightly by giving it the value 2 when $x = 1$ and calling the resulting function g :

$$g(x) = \begin{cases} \frac{x - 1}{x^2 - 1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

This new function g still has the same limit as x approaches 1. (See Figure 4.)

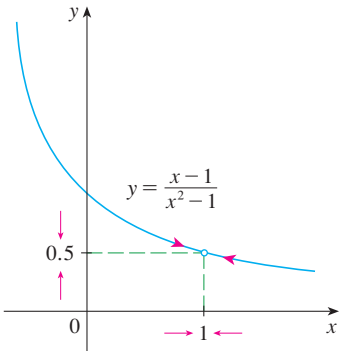


FIGURE 3

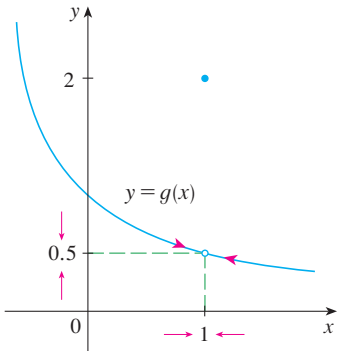


FIGURE 4

EXAMPLE 2 Estimate the value of $\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$.

SOLUTION The table lists values of the function for several values of t near 0.

t	$\frac{\sqrt{t^2 + 9} - 3}{t^2}$
± 1.0	0.16228
± 0.5	0.16553
± 0.1	0.16662
± 0.05	0.16666
± 0.01	0.16667

As t approaches 0, the values of the function seem to approach 0.1666666... and so we guess that

$$\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} = \frac{1}{6}$$

t	$\frac{\sqrt{t^2 + 9} - 3}{t^2}$
± 0.0005	0.16800
± 0.0001	0.20000
± 0.00005	0.00000
± 0.00001	0.00000

In Example 2 what would have happened if we had taken even smaller values of t ? The table in the margin shows the results from one calculator; you can see that something strange seems to be happening.

If you try these calculations on your own calculator you might get different values, but eventually you will get the value 0 if you make t sufficiently small. Does this mean that the answer is really 0 instead of $\frac{1}{6}$? No, the value of the limit is $\frac{1}{6}$, as we will show in the next section. The problem is that the **calculator gave false values** because $\sqrt{t^2 + 9}$ is very close to 3 when t is small. (In fact, when t is sufficiently small, a calculator's value for $\sqrt{t^2 + 9}$ is 3.000... to as many digits as the calculator is capable of carrying.)

Something similar happens when we try to graph the function

$$f(t) = \frac{\sqrt{t^2 + 9} - 3}{t^2}$$

of Example 2 on a graphing calculator or computer. Parts (a) and (b) of Figure 5 show quite accurate graphs of f , and when we use the trace mode (if available) we can estimate easily that the limit is about $\frac{1}{6}$. But if we zoom in too much, as in parts (c) and (d), then we get inaccurate graphs, again because of problems with subtraction.

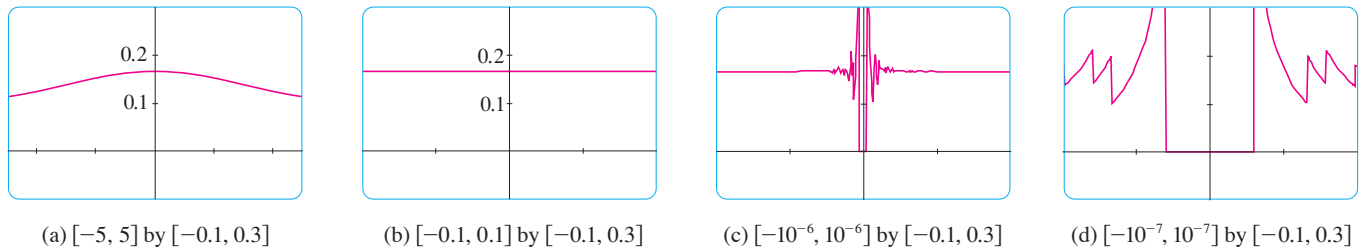


FIGURE 5

x	$\frac{\sin x}{x}$
± 1.0	0.84147098
± 0.5	0.95885108
± 0.4	0.97354586
± 0.3	0.98506736
± 0.2	0.99334665
± 0.1	0.99833417
± 0.05	0.99958339
± 0.01	0.99998333
± 0.005	0.99999583
± 0.001	0.99999983

V EXAMPLE 3 Guess the value of $\lim_{x \rightarrow 0} \frac{\sin x}{x}$.

SOLUTION The function $f(x) = (\sin x)/x$ is not defined when $x = 0$. Using a calculator (and remembering that, if $x \in \mathbb{R}$, $\sin x$ means the sine of the angle whose *radian* measure is x), we construct a table of values correct to eight decimal places. From the table at the left and the graph in Figure 6 we guess that

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

This guess is in fact correct, as will be proved in Chapter 3 using a geometric argument.

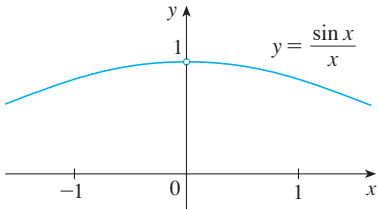


FIGURE 6

V EXAMPLE 4 Investigate $\lim_{x \rightarrow 0} \sin \frac{\pi}{x}$.

SOLUTION Again the function $f(x) = \sin(\pi/x)$ is undefined at 0. Evaluating the function for some small values of x , we get

$$\begin{aligned} f(1) &= \sin \pi = 0 & f\left(\frac{1}{2}\right) &= \sin 2\pi = 0 \\ f\left(\frac{1}{3}\right) &= \sin 3\pi = 0 & f\left(\frac{1}{4}\right) &= \sin 4\pi = 0 \\ f(0.1) &= \sin 10\pi = 0 & f(0.01) &= \sin 100\pi = 0 \end{aligned}$$

Similarly, $f(0.001) = f(0.0001) = 0$. On the basis of this information we might be tempted to guess that

$$\lim_{x \rightarrow 0} \sin \frac{\pi}{x} = 0$$

❗ but this time **our guess is wrong**. Note that although $f(1/n) = \sin n\pi = 0$ for any integer n , it is also true that $f(x) = 1$ for infinitely many values of x that approach 0. You can see this from the graph of f shown in Figure 7.

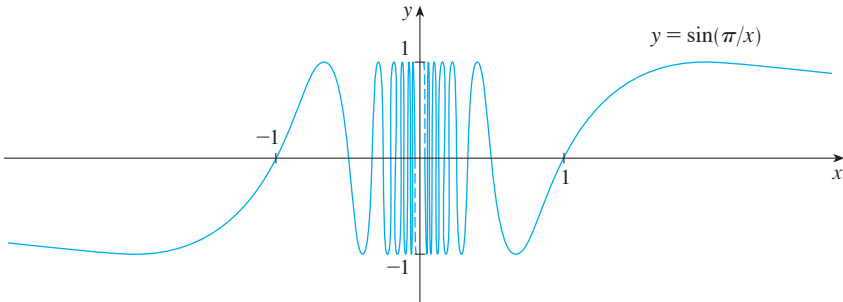


FIGURE 7

Computer Algebra Systems

Computer algebra systems (CAS) have commands that compute limits. In order to avoid the types of pitfalls demonstrated in Examples 2, 4, and 5, they don't find limits by numerical experimentation. Instead, they use more sophisticated techniques such as computing infinite series. If you have access to a CAS, use the limit command to compute the limits in the examples of this section and to check your answers in the exercises of this chapter.

The dashed lines near the y-axis indicate that the values of $\sin(\pi/x)$ oscillate between 1 and -1 infinitely often as x approaches 0. (See Exercise 45.)

Since the values of $f(x)$ do not approach a fixed number as x approaches 0,

$$\lim_{x \rightarrow 0} \sin \frac{\pi}{x} \text{ does not exist}$$

x	$x^3 + \frac{\cos 5x}{10,000}$
1	1.000028
0.5	0.124920
0.1	0.001088
0.05	0.000222
0.01	0.000101

x	$x^3 + \frac{\cos 5x}{10,000}$
0.005	0.00010009
0.001	0.00010000

EXAMPLE 5 Find $\lim_{x \rightarrow 0} \left(x^3 + \frac{\cos 5x}{10,000} \right)$.

SOLUTION As before, we construct a table of values. From the first table in the margin it appears that

$$\lim_{x \rightarrow 0} \left(x^3 + \frac{\cos 5x}{10,000} \right) = 0$$

But if we persevere with smaller values of x , the second table suggests that

$$\lim_{x \rightarrow 0} \left(x^3 + \frac{\cos 5x}{10,000} \right) = 0.000100 = \frac{1}{10,000}$$

Later we will see that $\lim_{x \rightarrow 0} \cos 5x = 1$; then it follows that the limit is 0.0001.

❗ Examples 4 and 5 illustrate some of the **pitfalls in guessing the value of a limit**. It is easy to guess the wrong value if we use inappropriate values of x , but it is difficult to know when to stop calculating values. And, as the discussion after Example 2 shows, sometimes calculators and computers give the wrong values. In the next section, however, we will develop foolproof methods for calculating limits.

V EXAMPLE 6 The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

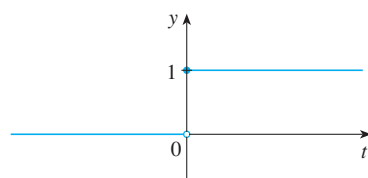


FIGURE 8
The Heaviside function

[This function is named after the electrical engineer Oliver Heaviside (1850–1925) and can be used to describe an electric current that is switched on at time $t = 0$.] Its graph is shown in Figure 8.

As t approaches 0 from the left, $H(t)$ approaches 0. As t approaches 0 from the right, $H(t)$ approaches 1. There is no single number that $H(t)$ approaches as t approaches 0. Therefore $\lim_{t \rightarrow 0} H(t)$ does not exist.

One-Sided Limits

We noticed in Example 6 that $H(t)$ approaches 0 as t approaches 0 from the left and $H(t)$ approaches 1 as t approaches 0 from the right. We indicate this situation symbolically by writing

$$\lim_{t \rightarrow 0^-} H(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow 0^+} H(t) = 1$$

The symbol “ $t \rightarrow 0^-$ ” indicates that we consider only values of t that are less than 0. Likewise, “ $t \rightarrow 0^+$ ” indicates that we consider only values of t that are greater than 0.

2 Definition We write

$$\lim_{x \rightarrow a^-} f(x) = L$$

and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a .

Notice that Definition 2 differs from Definition 1 only in that we require x to be less than a . Similarly, if we require that x be greater than a , we get “the **right-hand limit of $f(x)$ as x approaches a** is equal to L ” and we write

$$\lim_{x \rightarrow a^+} f(x) = L$$

Thus the symbol “ $x \rightarrow a^+$ ” means that we consider only $x > a$. These definitions are illustrated in Figure 9.

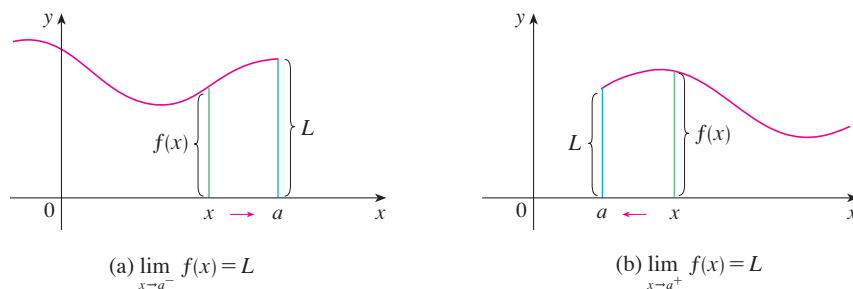


FIGURE 9

By comparing Definition 1 with the definitions of one-sided limits, we see that the following is true.

3 $\lim_{x \rightarrow a} f(x) = L$ if and only if $\lim_{x \rightarrow a^-} f(x) = L$ and $\lim_{x \rightarrow a^+} f(x) = L$

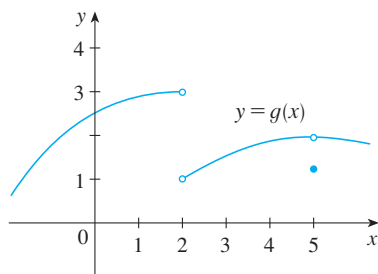


FIGURE 10

V EXAMPLE 7 The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

- (a) $\lim_{x \rightarrow 2^-} g(x)$ (b) $\lim_{x \rightarrow 2^+} g(x)$ (c) $\lim_{x \rightarrow 2} g(x)$
 (d) $\lim_{x \rightarrow 5^-} g(x)$ (e) $\lim_{x \rightarrow 5^+} g(x)$ (f) $\lim_{x \rightarrow 5} g(x)$

SOLUTION From the graph we see that the values of $g(x)$ approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right. Therefore

$$(a) \lim_{x \rightarrow 2^-} g(x) = 3 \quad \text{and} \quad (b) \lim_{x \rightarrow 2^+} g(x) = 1$$

(c) Since the left and right limits are different, we conclude from [3] that $\lim_{x \rightarrow 2} g(x)$ does not exist.

The graph also shows that

$$(d) \lim_{x \rightarrow 5^-} g(x) = 2 \quad \text{and} \quad (e) \lim_{x \rightarrow 5^+} g(x) = 2$$

(f) This time the left and right limits are the same and so, by [3], we have

$$\lim_{x \rightarrow 5} g(x) = 2$$

Despite this fact, notice that $g(5) \neq 2$.

Infinite Limits

EXAMPLE 8 Find $\lim_{x \rightarrow 0} \frac{1}{x^2}$ if it exists.

SOLUTION As x becomes close to 0, x^2 also becomes close to 0, and $1/x^2$ becomes very large. (See the table in the margin.) In fact, it appears from the graph of the function $f(x) = 1/x^2$ shown in Figure 11 that the values of $f(x)$ can be made arbitrarily large by taking x close enough to 0. Thus the values of $f(x)$ do not approach a number, so $\lim_{x \rightarrow 0} (1/x^2)$ does not exist.

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

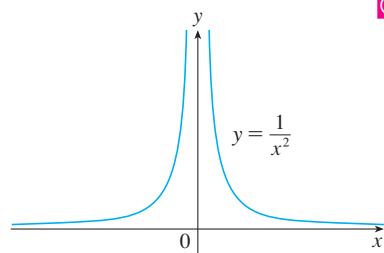


FIGURE 11

To indicate the kind of behavior exhibited in Example 8, we use the notation

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$

⊗ This does not mean that we are regarding ∞ as a number. Nor does it mean that the limit exists. It simply expresses the particular way in which the limit does not exist: $1/x^2$ can be made as large as we like by taking x close enough to 0.

In general, we write symbolically

$$\lim_{x \rightarrow a} f(x) = \infty$$

to indicate that the values of $f(x)$ tend to become larger and larger (or “increase without bound”) as x becomes closer and closer to a .

4 Definition Let f be a function defined on both sides of a , except possibly at a itself. Then

$$\lim_{x \rightarrow a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a , but not equal to a .

Another notation for $\lim_{x \rightarrow a} f(x) = \infty$ is

$$f(x) \rightarrow \infty \quad \text{as} \quad x \rightarrow a$$

Again, the symbol ∞ is not a number, but the expression $\lim_{x \rightarrow a} f(x) = \infty$ is often read as

“the limit of $f(x)$, as x approaches a , is infinity”

or

“ $f(x)$ becomes infinite as x approaches a ”

or

“ $f(x)$ increases without bound as x approaches a ”

This definition is illustrated graphically in Figure 12.

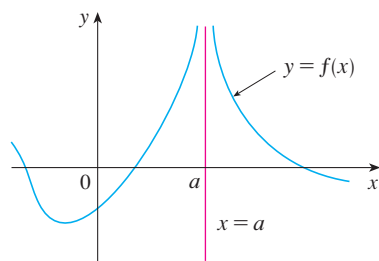


FIGURE 12

$$\lim_{x \rightarrow a} f(x) = \infty$$

When we say a number is “large negative,” we mean that it is negative but its magnitude (absolute value) is large.

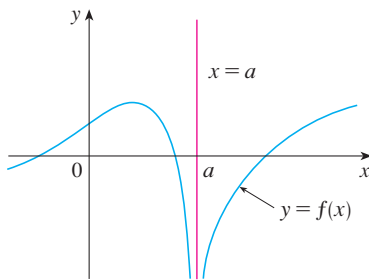


FIGURE 13

$$\lim_{x \rightarrow a} f(x) = -\infty$$

A similar sort of limit, for functions that become large negative as x gets close to a , is defined in Definition 5 and is illustrated in Figure 13.

5 Definition Let f be defined on both sides of a , except possibly at a itself. Then

$$\lim_{x \rightarrow a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a , but not equal to a .

The symbol $\lim_{x \rightarrow a} f(x) = -\infty$ can be read as “the limit of $f(x)$, as x approaches a , is negative infinity” or “ $f(x)$ decreases without bound as x approaches a .” As an example we have

$$\lim_{x \rightarrow 0} \left(-\frac{1}{x^2} \right) = -\infty$$

Similar definitions can be given for the one-sided infinite limits

$$\lim_{x \rightarrow a^-} f(x) = \infty$$

$$\lim_{x \rightarrow a^+} f(x) = \infty$$

$$\lim_{x \rightarrow a^-} f(x) = -\infty$$

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$

remembering that “ $x \rightarrow a^-$ ” means that we consider only values of x that are less than a , and similarly “ $x \rightarrow a^+$ ” means that we consider only $x > a$. Illustrations of these four cases are given in Figure 14.

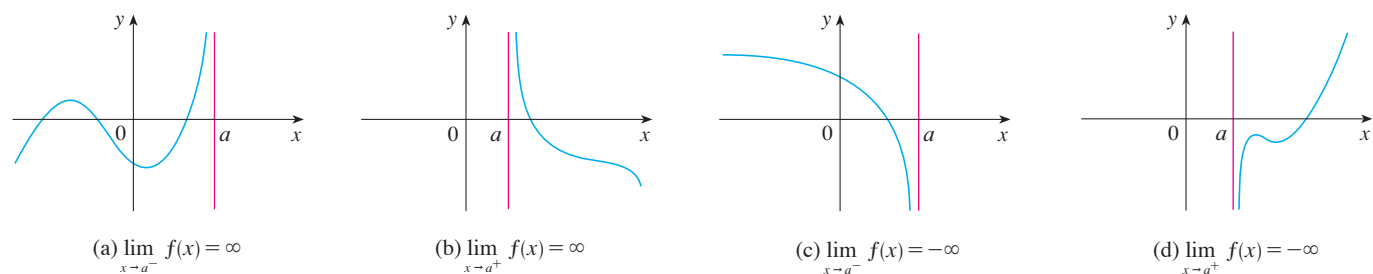


FIGURE 14

6 Definition The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \rightarrow a^-} f(x) = \infty$$

$$\lim_{x \rightarrow a^-} f(x) = \infty$$

$$\lim_{x \rightarrow a^+} f(x) = \infty$$

$$\lim_{x \rightarrow a^-} f(x) = -\infty$$

$$\lim_{x \rightarrow a^-} f(x) = -\infty$$

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$

For instance, the y -axis is a vertical asymptote of the curve $y = 1/x^2$ because $\lim_{x \rightarrow 0} (1/x^2) = \infty$. In Figure 14 the line $x = a$ is a vertical asymptote in each of the four cases shown. In general, knowledge of vertical asymptotes is very useful in sketching graphs.

EXAMPLE 9 Find $\lim_{x \rightarrow 3^+} \frac{2x}{x-3}$ and $\lim_{x \rightarrow 3^-} \frac{2x}{x-3}$.

SOLUTION If x is close to 3 but larger than 3, then the denominator $x - 3$ is a small positive number and $2x$ is close to 6. So the quotient $2x/(x - 3)$ is a large *positive* number. Thus, intuitively, we see that

$$\lim_{x \rightarrow 3^+} \frac{2x}{x-3} = \infty$$

Likewise, if x is close to 3 but smaller than 3, then $x - 3$ is a small negative number but $2x$ is still a positive number (close to 6). So $2x/(x - 3)$ is a numerically large *negative* number. Thus

$$\lim_{x \rightarrow 3^-} \frac{2x}{x-3} = -\infty$$

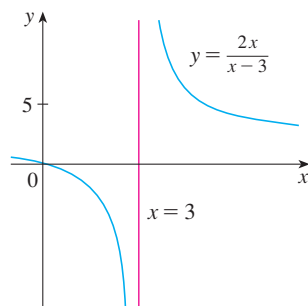


FIGURE 15

The graph of the curve $y = 2x/(x - 3)$ is given in Figure 15. The line $x = 3$ is a vertical asymptote.

EXAMPLE 10 Find the vertical asymptotes of $f(x) = \tan x$.

SOLUTION Because

$$\tan x = \frac{\sin x}{\cos x}$$

there are potential vertical asymptotes where $\cos x = 0$. In fact, since $\cos x \rightarrow 0^+$ as $x \rightarrow (\pi/2)^-$ and $\cos x \rightarrow 0^-$ as $x \rightarrow (\pi/2)^+$, whereas $\sin x$ is positive when x is near $\pi/2$, we have

$$\lim_{x \rightarrow (\pi/2)^-} \tan x = \infty \quad \text{and} \quad \lim_{x \rightarrow (\pi/2)^+} \tan x = -\infty$$

This shows that the line $x = \pi/2$ is a vertical asymptote. Similar reasoning shows that the lines $x = (2n + 1)\pi/2$, where n is an integer, are all vertical asymptotes of $f(x) = \tan x$. The graph in Figure 16 confirms this.

Another example of a function whose graph has a vertical asymptote is the natural logarithmic function $y = \ln x$. From Figure 17 we see that

$$\lim_{x \rightarrow 0^+} \ln x = -\infty$$

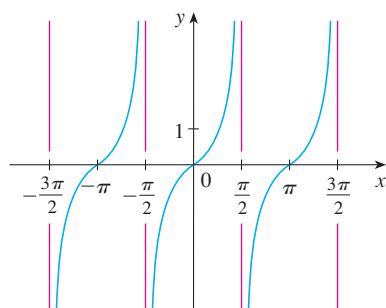


FIGURE 16

$y = \tan x$

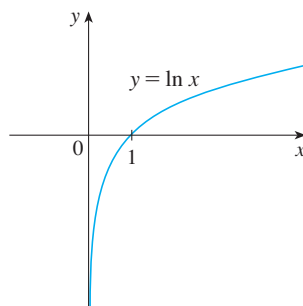


FIGURE 17

The y -axis is a vertical asymptote of the natural logarithmic function.

and so the line $x = 0$ (the y -axis) is a vertical asymptote. In fact, the same is true for $y = \log_a x$ provided that $a > 1$. (See Figures 11 and 12 in Section 1.6.)

2.2 Exercises

1. Explain in your own words what is meant by the equation

$$\lim_{x \rightarrow 2} f(x) = 5$$

Is it possible for this statement to be true and yet $f(2) = 3$? Explain.

2. Explain what it means to say that

$$\lim_{x \rightarrow 1^-} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = 7$$

In this situation is it possible that $\lim_{x \rightarrow 1} f(x)$ exists? Explain.

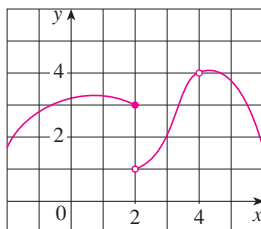
3. Explain the meaning of each of the following.

$$(a) \lim_{x \rightarrow -3} f(x) = \infty \quad (b) \lim_{x \rightarrow 4^+} f(x) = -\infty$$

4. Use the given graph of
- f
- to state the value of each quantity, if it exists. If it does not exist, explain why.

$$(a) \lim_{x \rightarrow 2^-} f(x) \quad (b) \lim_{x \rightarrow 2^+} f(x) \quad (c) \lim_{x \rightarrow 2} f(x)$$

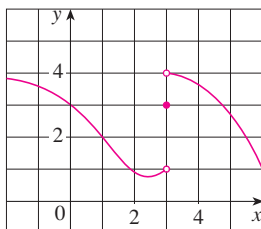
$$(d) f(2) \quad (e) \lim_{x \rightarrow 4} f(x) \quad (f) f(4)$$



5. For the function
- f
- whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

$$(a) \lim_{x \rightarrow 1} f(x) \quad (b) \lim_{x \rightarrow 3^-} f(x) \quad (c) \lim_{x \rightarrow 3^+} f(x)$$

$$(d) \lim_{x \rightarrow 3} f(x) \quad (e) f(3)$$



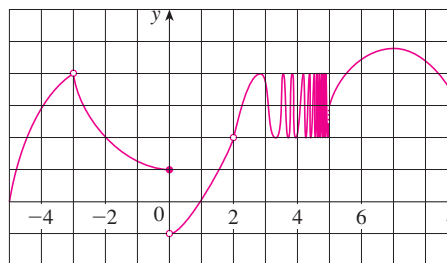
6. For the function
- h
- whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

$$(a) \lim_{x \rightarrow -3^-} h(x) \quad (b) \lim_{x \rightarrow -3^+} h(x) \quad (c) \lim_{x \rightarrow -3} h(x)$$

$$(d) h(-3) \quad (e) \lim_{x \rightarrow 0^-} h(x) \quad (f) \lim_{x \rightarrow 0^+} h(x)$$

$$(g) \lim_{x \rightarrow 0} h(x) \quad (h) h(0) \quad (i) \lim_{x \rightarrow 2} h(x)$$

$$(j) h(2) \quad (k) \lim_{x \rightarrow 5^+} h(x) \quad (l) \lim_{x \rightarrow 5^-} h(x)$$

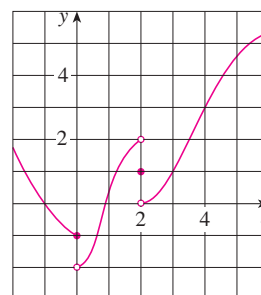


7. For the function
- g
- whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

$$(a) \lim_{t \rightarrow 0^-} g(t) \quad (b) \lim_{t \rightarrow 0^+} g(t) \quad (c) \lim_{t \rightarrow 0} g(t)$$

$$(d) \lim_{t \rightarrow 2^-} g(t) \quad (e) \lim_{t \rightarrow 2^+} g(t) \quad (f) \lim_{t \rightarrow 2} g(t)$$

$$(g) g(2) \quad (h) \lim_{t \rightarrow 4} g(t)$$

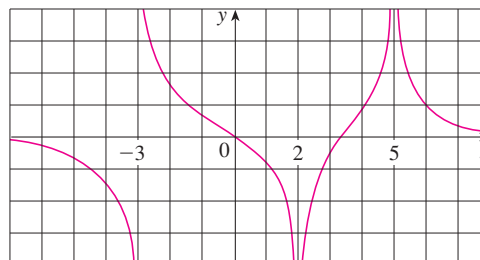


8. For the function
- R
- whose graph is shown, state the following.

$$(a) \lim_{x \rightarrow 2} R(x) \quad (b) \lim_{x \rightarrow 5} R(x)$$

$$(c) \lim_{x \rightarrow -3^-} R(x) \quad (d) \lim_{x \rightarrow -3^+} R(x)$$

(e) The equations of the vertical asymptotes.

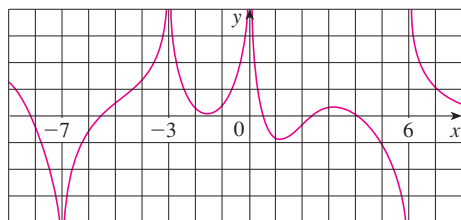


9. For the function f whose graph is shown, state the following.

(a) $\lim_{x \rightarrow -7} f(x)$ (b) $\lim_{x \rightarrow -3} f(x)$ (c) $\lim_{x \rightarrow 0} f(x)$

(d) $\lim_{x \rightarrow 6^-} f(x)$ (e) $\lim_{x \rightarrow 6^+} f(x)$

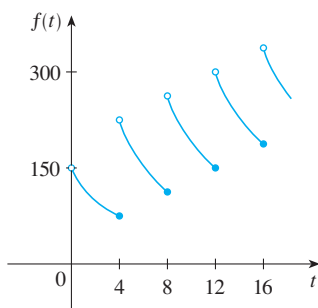
- (f) The equations of the vertical asymptotes.



10. A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the bloodstream after t hours. Find

$$\lim_{t \rightarrow 12^-} f(t) \quad \text{and} \quad \lim_{t \rightarrow 12^+} f(t)$$

and explain the significance of these one-sided limits.



- 11–12 Sketch the graph of the function and use it to determine the values of a for which $\lim_{x \rightarrow a} f(x)$ exists.

$$11. f(x) = \begin{cases} 1 + x & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x < 1 \\ 2 - x & \text{if } x \geq 1 \end{cases}$$

$$12. f(x) = \begin{cases} 1 + \sin x & \text{if } x < 0 \\ \cos x & \text{if } 0 \leq x \leq \pi \\ \sin x & \text{if } x > \pi \end{cases}$$

- 13–14 Use the graph of the function f to state the value of each limit, if it exists. If it does not exist, explain why.

(a) $\lim_{x \rightarrow 0^-} f(x)$ (b) $\lim_{x \rightarrow 0^+} f(x)$ (c) $\lim_{x \rightarrow 0} f(x)$

13. $f(x) = \frac{1}{1 + e^{1/x}}$ 14. $f(x) = \frac{x^2 + x}{\sqrt{x^3 + x^2}}$

- 15–18 Sketch the graph of an example of a function f that satisfies all of the given conditions.

15. $\lim_{x \rightarrow 0^-} f(x) = -1$, $\lim_{x \rightarrow 0^+} f(x) = 2$, $f(0) = 1$

16. $\lim_{x \rightarrow 0} f(x) = 1$, $\lim_{x \rightarrow 3^-} f(x) = -2$, $\lim_{x \rightarrow 3^+} f(x) = 2$,
 $f(0) = -1$, $f(3) = 1$

17. $\lim_{x \rightarrow 3^+} f(x) = 4$, $\lim_{x \rightarrow 3^-} f(x) = 2$, $\lim_{x \rightarrow -2} f(x) = 2$,
 $f(3) = 3$, $f(-2) = 1$

18. $\lim_{x \rightarrow 0^-} f(x) = 2$, $\lim_{x \rightarrow 0^+} f(x) = 0$, $\lim_{x \rightarrow 4^-} f(x) = 3$,
 $\lim_{x \rightarrow 4^+} f(x) = 0$, $f(0) = 2$, $f(4) = 1$

- 19–22 Guess the value of the limit (if it exists) by evaluating the function at the given numbers (correct to six decimal places).

19. $\lim_{x \rightarrow 2} \frac{x^2 - 2x}{x^2 - x - 2}$,
 $x = 2.5, 2.1, 2.05, 2.01, 2.005, 2.001,$
 $1.9, 1.95, 1.99, 1.995, 1.999$

20. $\lim_{x \rightarrow -1} \frac{x^2 - 2x}{x^2 - x - 2}$,
 $x = 0, -0.5, -0.9, -0.95, -0.99, -0.999,$
 $-2, -1.5, -1.1, -1.01, -1.001$

21. $\lim_{t \rightarrow 0} \frac{e^{5t} - 1}{t}$, $t = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$

22. $\lim_{h \rightarrow 0} \frac{(2 + h)^5 - 32}{h}$,
 $h = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$

- 23–26 Use a table of values to estimate the value of the limit. If you have a graphing device, use it to confirm your result graphically.

23. $\lim_{x \rightarrow 0} \frac{\sqrt{x+4} - 2}{x}$

24. $\lim_{x \rightarrow 0} \frac{\tan 3x}{\tan 5x}$

25. $\lim_{x \rightarrow 1} \frac{x^6 - 1}{x^{10} - 1}$

26. $\lim_{x \rightarrow 0} \frac{9^x - 5^x}{x}$

27. (a) By graphing the function $f(x) = (\cos 2x - \cos x)/x^2$ and zooming in toward the point where the graph crosses the y -axis, estimate the value of $\lim_{x \rightarrow 0} f(x)$.
(b) Check your answer in part (a) by evaluating $f(x)$ for values of x that approach 0.

-  28. (a) Estimate the value of

$$\lim_{x \rightarrow 0} \frac{\sin x}{\sin \pi x}$$

by graphing the function $f(x) = (\sin x)/(\sin \pi x)$. State your answer correct to two decimal places.

- (b) Check your answer in part (a) by evaluating $f(x)$ for values of x that approach 0.

29–37 Determine the infinite limit.

29. $\lim_{x \rightarrow -3^+} \frac{x+2}{x+3}$

30. $\lim_{x \rightarrow -3^-} \frac{x+2}{x+3}$

31. $\lim_{x \rightarrow 1} \frac{2-x}{(x-1)^2}$

32. $\lim_{x \rightarrow 5^-} \frac{e^x}{(x-5)^3}$

33. $\lim_{x \rightarrow 3^+} \ln(x^2 - 9)$

34. $\lim_{x \rightarrow \pi^-} \cot x$


35. $\lim_{x \rightarrow 2\pi^-} x \csc x$

36. $\lim_{x \rightarrow 2^-} \frac{x^2 - 2x}{x^2 - 4x + 4}$

37. $\lim_{x \rightarrow 2^+} \frac{x^2 - 2x - 8}{x^2 - 5x + 6}$


38. (a) Find the vertical asymptotes of the function

$$y = \frac{x^2 + 1}{3x - 2x^2}$$

-  (b) Confirm your answer to part (a) by graphing the function.

39. Determine $\lim_{x \rightarrow 1^-} \frac{1}{x^3 - 1}$ and $\lim_{x \rightarrow 1^+} \frac{1}{x^3 - 1}$


- (a) by evaluating $f(x) = 1/(x^3 - 1)$ for values of x that approach 1 from the left and from the right,
 (b) by reasoning as in Example 9, and
 (c) from a graph of f .

-  40. (a) By graphing the function $f(x) = (\tan 4x)/x$ and zooming in toward the point where the graph crosses the y -axis, estimate the value of $\lim_{x \rightarrow 0} f(x)$.

- (b) Check your answer in part (a) by evaluating $f(x)$ for values of x that approach 0.

41. (a) Estimate the value of the limit $\lim_{x \rightarrow 0} (1 + x)^{1/x}$ to five decimal places. Does this number look familiar?

-  (b) Illustrate part (a) by graphing the function $y = (1 + x)^{1/x}$.

-  42. (a) Graph the function $f(x) = e^x + \ln|x - 4|$ for $0 \leq x \leq 5$. Do you think the graph is an accurate representation of f ?

- (b) How would you get a graph that represents f better?

43. (a) Evaluate the function $f(x) = x^2 - (2^x/1000)$ for $x = 1, 0.8, 0.6, 0.4, 0.2, 0.1$, and 0.05 , and guess the value of


$$\lim_{x \rightarrow 0} \left(x^2 - \frac{2^x}{1000} \right)$$


- (b) Evaluate $f(x)$ for $x = 0.04, 0.02, 0.01, 0.005, 0.003$, and 0.001 . Guess again.

44. (a) Evaluate $h(x) = (\tan x - x)/x^3$ for $x = 1, 0.5, 0.1, 0.05, 0.01$, and 0.005 .

- (b) Guess the value of $\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$.

- (c) Evaluate $h(x)$ for successively smaller values of x until you finally reach a value of 0 for $h(x)$. Are you still confident that your guess in part (b) is correct? Explain why you eventually obtained 0 values. (In Section 4.4 a method for evaluating the limit will be explained.)


-  (d) Graph the function h in the viewing rectangle $[-1, 1]$ by $[0, 1]$. Then zoom in toward the point where the graph crosses the y -axis to estimate the limit of $h(x)$ as x approaches 0. Continue to zoom in until you observe distortions in the graph of h . Compare with the results of part (c).

-  45. Graph the function $f(x) = \sin(\pi/x)$ of Example 4 in the viewing rectangle $[-1, 1]$ by $[-1, 1]$. Then zoom in toward the origin several times. Comment on the behavior of this function.

46. In the theory of relativity, the mass of a particle with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the particle at rest and c is the speed of light. What happens as $v \rightarrow c^-$?

-  47. Use a graph to estimate the equations of all the vertical asymptotes of the curve

$$y = \tan(2 \sin x) \quad -\pi \leq x \leq \pi$$

Then find the exact equations of these asymptotes.

-  48. (a) Use numerical and graphical evidence to guess the value of the limit

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{\sqrt{x} - 1}$$

- (b) How close to 1 does x have to be to ensure that the function in part (a) is within a distance 0.5 of its limit?

2.3 Calculating Limits Using the Limit Laws

In Section 2.2 we used calculators and graphs to guess the values of limits, but we saw that such methods don't always lead to the correct answer. In this section we use the following properties of limits, called the *Limit Laws*, to calculate limits.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \rightarrow a} f(x) \quad \text{and} \quad \lim_{x \rightarrow a} g(x)$$

exist. Then

$$1. \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

$$2. \lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

$$3. \lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$$

$$4. \lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

$$5. \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{if } \lim_{x \rightarrow a} g(x) \neq 0$$

These five laws can be stated verbally as follows:

1. The limit of a sum is the sum of the limits.
2. The limit of a difference is the difference of the limits.
3. The limit of a constant times a function is the constant times the limit of the function.
4. The limit of a product is the product of the limits.
5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

It is easy to believe that these properties are true. For instance, if $f(x)$ is close to L and $g(x)$ is close to M , it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$. This gives us an intuitive basis for believing that Law 1 is true. In Section 2.4 we give a precise definition of a limit and use it to prove this law. The proofs of the remaining laws are given in Appendix F.

Sum Law

Difference Law

Constant Multiple Law

Product Law

Quotient Law

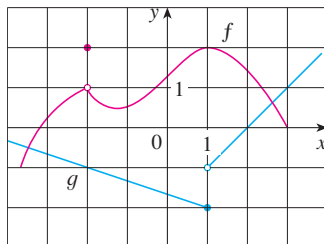


FIGURE 1

EXAMPLE 1 Use the Limit Laws and the graphs of f and g in Figure 1 to evaluate the following limits, if they exist.

$$(a) \lim_{x \rightarrow -2} [f(x) + 5g(x)] \quad (b) \lim_{x \rightarrow 1} [f(x)g(x)] \quad (c) \lim_{x \rightarrow 2} \frac{f(x)}{g(x)}$$

SOLUTION

(a) From the graphs of f and g we see that

$$\lim_{x \rightarrow -2} f(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow -2} g(x) = -1$$

Therefore we have

$$\begin{aligned}\lim_{x \rightarrow -2} [f(x) + 5g(x)] &= \lim_{x \rightarrow -2} f(x) + \lim_{x \rightarrow -2} [5g(x)] && \text{(by Law 1)} \\ &= \lim_{x \rightarrow -2} f(x) + 5 \lim_{x \rightarrow -2} g(x) && \text{(by Law 3)} \\ &= 1 + 5(-1) = -4\end{aligned}$$

(b) We see that $\lim_{x \rightarrow 1} f(x) = 2$. But $\lim_{x \rightarrow 1} g(x)$ does not exist because the left and right limits are different:

$$\lim_{x \rightarrow 1^-} g(x) = -2 \qquad \lim_{x \rightarrow 1^+} g(x) = -1$$

So we can't use Law 4 for the desired limit. But we *can* use Law 4 for the one-sided limits:

$$\lim_{x \rightarrow 1^-} [f(x)g(x)] = 2 \cdot (-2) = -4 \qquad \lim_{x \rightarrow 1^+} [f(x)g(x)] = 2 \cdot (-1) = -2$$

The left and right limits aren't equal, so $\lim_{x \rightarrow 1} [f(x)g(x)]$ does not exist.

(c) The graphs show that

$$\lim_{x \rightarrow 2} f(x) \approx 1.4 \qquad \text{and} \qquad \lim_{x \rightarrow 2} g(x) = 0$$

Because the limit of the denominator is 0, we can't use Law 5. The given limit does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

If we use the Product Law repeatedly with $g(x) = f(x)$, we obtain the following law.

Power Law

$$6. \lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n \quad \text{where } n \text{ is a positive integer}$$

In applying these six limit laws, we need to use two special limits:

$$7. \lim_{x \rightarrow a} c = c$$

$$8. \lim_{x \rightarrow a} x = a$$

These limits are obvious from an intuitive point of view (state them in words or draw graphs of $y = c$ and $y = x$), but proofs based on the precise definition are requested in the exercises for Section 2.4.

If we now put $f(x) = x$ in Law 6 and use Law 8, we get another useful special limit.

$$9. \lim_{x \rightarrow a} x^n = a^n \quad \text{where } n \text{ is a positive integer}$$

A similar limit holds for roots as follows. (For square roots the proof is outlined in Exercise 37 in Section 2.4.)

$$10. \lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a} \quad \text{where } n \text{ is a positive integer}$$

(If n is even, we assume that $a > 0$.)

More generally, we have the following law, which is proved in Section 2.5 as a consequence of Law 10.

Root Law

$$11. \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} \quad \text{where } n \text{ is a positive integer}$$

[If n is even, we assume that $\lim_{x \rightarrow a} f(x) > 0$.]

Newton and Limits

Isaac Newton was born on Christmas Day in 1642, the year of Galileo's death. When he entered Cambridge University in 1661 Newton didn't know much mathematics, but he learned quickly by reading Euclid and Descartes and by attending the lectures of Isaac Barrow. Cambridge was closed because of the plague in 1665 and 1666, and Newton returned home to reflect on what he had learned. Those two years were amazingly productive for at that time he made four of his major discoveries: (1) his representation of functions as sums of infinite series, including the binomial theorem; (2) his work on differential and integral calculus; (3) his laws of motion and law of universal gravitation; and (4) his prism experiments on the nature of light and color. Because of a fear of controversy and criticism, he was reluctant to publish his discoveries and it wasn't until 1687, at the urging of the astronomer Halley, that Newton published *Principia Mathematica*. In this work, the greatest scientific treatise ever written, Newton set forth his version of calculus and used it to investigate mechanics, fluid dynamics, and wave motion, and to explain the motion of planets and comets.

The beginnings of calculus are found in the calculations of areas and volumes by ancient Greek scholars such as Eudoxus and Archimedes. Although aspects of the idea of a limit are implicit in their "method of exhaustion," Eudoxus and Archimedes never explicitly formulated the concept of a limit. Likewise, mathematicians such as Cavalieri, Fermat, and Barrow, the immediate precursors of Newton in the development of calculus, did not actually use limits. It was Isaac Newton who was the first to talk explicitly about limits. He explained that the main idea behind limits is that quantities "approach nearer than by any given difference." Newton stated that the limit was the basic concept in calculus, but it was left to later mathematicians like Cauchy to clarify his ideas about limits.

EXAMPLE 2 Evaluate the following limits and justify each step.

$$(a) \lim_{x \rightarrow 5} (2x^2 - 3x + 4) \qquad (b) \lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

SOLUTION

$$(a) \lim_{x \rightarrow 5} (2x^2 - 3x + 4) = \lim_{x \rightarrow 5} (2x^2) - \lim_{x \rightarrow 5} (3x) + \lim_{x \rightarrow 5} 4 \quad (\text{by Laws 2 and 1})$$

$$= 2 \lim_{x \rightarrow 5} x^2 - 3 \lim_{x \rightarrow 5} x + \lim_{x \rightarrow 5} 4 \quad (\text{by 3})$$

$$= 2(5^2) - 3(5) + 4 \quad (\text{by 9, 8, and 7})$$

$$= 39$$

(b) We start by using Law 5, but its use is fully justified only at the final stage when we see that the limits of the numerator and denominator exist and the limit of the denominator is not 0.

$$\lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{\lim_{x \rightarrow -2} (x^3 + 2x^2 - 1)}{\lim_{x \rightarrow -2} (5 - 3x)} \quad (\text{by Law 5})$$

$$= \frac{\lim_{x \rightarrow -2} x^3 + 2 \lim_{x \rightarrow -2} x^2 - \lim_{x \rightarrow -2} 1}{\lim_{x \rightarrow -2} 5 - 3 \lim_{x \rightarrow -2} x} \quad (\text{by 1, 2, and 3})$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} \quad (\text{by 9, 8, and 7})$$

$$= -\frac{1}{11}$$

NOTE If we let $f(x) = 2x^2 - 3x + 4$, then $f(5) = 39$. In other words, we would have gotten the correct answer in Example 2(a) by substituting 5 for x . Similarly, direct substitution provides the correct answer in part (b). The functions in Example 2 are a polynomial and a rational function, respectively, and similar use of the Limit Laws proves that direct substitution always works for such functions (see Exercises 55 and 56). We state this fact as follows.

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Functions with the Direct Substitution Property are called *continuous at a* and will be studied in Section 2.5. However, not all limits can be evaluated by direct substitution, as the following examples show.

EXAMPLE 3 Find $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$.

SOLUTION Let $f(x) = (x^2 - 1)/(x - 1)$. We can't find the limit by substituting $x = 1$ because $f(1)$ isn't defined. Nor can we apply the Quotient Law, because the limit of the denominator is 0. Instead, we need to do some preliminary algebra. We factor the numerator as a difference of squares:

$$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1}$$

The numerator and denominator have a common factor of $x - 1$. When we take the limit as x approaches 1, we have $x \neq 1$ and so $x - 1 \neq 0$. Therefore we can cancel the common factor and compute the limit as follows:

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{x - 1} \\ &= \lim_{x \rightarrow 1} (x + 1) \\ &= 1 + 1 = 2 \end{aligned}$$

The limit in this example arose in Section 2.1 when we were trying to find the tangent to the parabola $y = x^2$ at the point $(1, 1)$.

NOTE In Example 3 we were able to compute the limit by replacing the given function $f(x) = (x^2 - 1)/(x - 1)$ by a simpler function, $g(x) = x + 1$, with the same limit. This is valid because $f(x) = g(x)$ except when $x = 1$, and in computing a limit as x approaches 1 we don't consider what happens when x is actually *equal* to 1. In general, we have the following useful fact.

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$, provided the limits exist.

EXAMPLE 4 Find $\lim_{x \rightarrow 1} g(x)$ where

$$g(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases}$$

SOLUTION Here g is defined at $x = 1$ and $g(1) = \pi$, but the value of a limit as x approaches 1 does not depend on the value of the function at 1. Since $g(x) = x + 1$ for $x \neq 1$, we have

$$\lim_{x \rightarrow 1} g(x) = \lim_{x \rightarrow 1} (x + 1) = 2$$

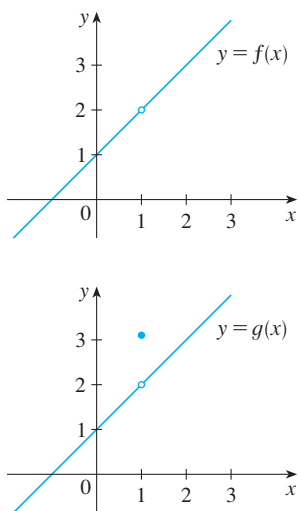


FIGURE 2

The graphs of the functions f (from Example 3) and g (from Example 4)

Note that the values of the functions in Examples 3 and 4 are identical except when $x = 1$ (see Figure 2) and so they have the same limit as x approaches 1.

V EXAMPLE 5 Evaluate $\lim_{h \rightarrow 0} \frac{(3 + h)^2 - 9}{h}$.

SOLUTION If we define

$$F(h) = \frac{(3 + h)^2 - 9}{h}$$

then, as in Example 3, we can't compute $\lim_{h \rightarrow 0} F(h)$ by letting $h = 0$ since $F(0)$ is undefined. But if we simplify $F(h)$ algebraically, we find that

$$F(h) = \frac{(9 + 6h + h^2) - 9}{h} = \frac{6h + h^2}{h} = 6 + h$$

(Recall that we consider only $h \neq 0$ when letting h approach 0.) Thus

$$\lim_{h \rightarrow 0} \frac{(3 + h)^2 - 9}{h} = \lim_{h \rightarrow 0} (6 + h) = 6$$

EXAMPLE 6 Find $\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$.

SOLUTION We can't apply the Quotient Law immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator:

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} &= \lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} \cdot \frac{\sqrt{t^2 + 9} + 3}{\sqrt{t^2 + 9} + 3} \\ &= \lim_{t \rightarrow 0} \frac{(t^2 + 9) - 9}{t^2(\sqrt{t^2 + 9} + 3)} \\ &= \lim_{t \rightarrow 0} \frac{t^2}{t^2(\sqrt{t^2 + 9} + 3)} \\ &= \lim_{t \rightarrow 0} \frac{1}{\sqrt{t^2 + 9} + 3} \\ &= \frac{1}{\sqrt{\lim_{t \rightarrow 0} (t^2 + 9)} + 3} \\ &= \frac{1}{3 + 3} = \frac{1}{6} \end{aligned}$$

This calculation confirms the guess that we made in Example 2 in Section 2.2.

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem is a reminder of what we discovered in Section 2.2. It says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

1 Theorem $\lim_{x \rightarrow a} f(x) = L$ if and only if $\lim_{x \rightarrow a^-} f(x) = L = \lim_{x \rightarrow a^+} f(x)$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

EXAMPLE 7 Show that $\lim_{x \rightarrow 0} |x| = 0$.

SOLUTION Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Since $|x| = x$ for $x > 0$, we have

$$\lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ and so

$$\lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^-} (-x) = 0$$

Therefore, by Theorem 1,

$$\lim_{x \rightarrow 0} |x| = 0$$

The result of Example 7 looks plausible from Figure 3.

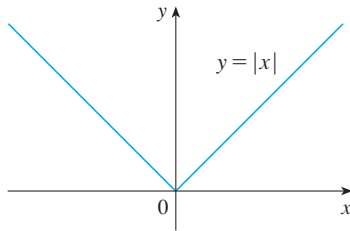


FIGURE 3

EXAMPLE 8 Prove that $\lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist.

SOLUTION

$$\lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} \frac{x}{x} = \lim_{x \rightarrow 0^+} 1 = 1$$

$$\lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = \lim_{x \rightarrow 0^-} (-1) = -1$$

Since the right- and left-hand limits are different, it follows from Theorem 1 that $\lim_{x \rightarrow 0} |x|/x$ does not exist. The graph of the function $f(x) = |x|/x$ is shown in Figure 4 and supports the one-sided limits that we found.

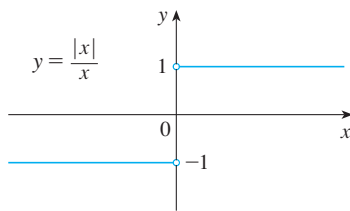


FIGURE 4

EXAMPLE 9 If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8-2x & \text{if } x < 4 \end{cases}$$

determine whether $\lim_{x \rightarrow 4} f(x)$ exists.

SOLUTION Since $f(x) = \sqrt{x-4}$ for $x > 4$, we have

$$\lim_{x \rightarrow 4^+} f(x) = \lim_{x \rightarrow 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$$

Since $f(x) = 8 - 2x$ for $x < 4$, we have

$$\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^-} (8 - 2x) = 8 - 2 \cdot 4 = 0$$

The right- and left-hand limits are equal. Thus the limit exists and

$$\lim_{x \rightarrow 4} f(x) = 0$$

The graph of f is shown in Figure 5.

It is shown in Example 3 in Section 2.4 that $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$.

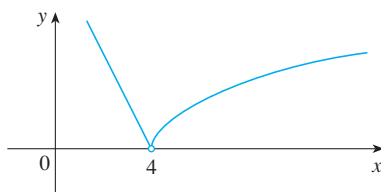


FIGURE 5

Other notations for $\llbracket x \rrbracket$ are $[x]$ and $\lfloor x \rfloor$. The greatest integer function is sometimes called the *floor function*.

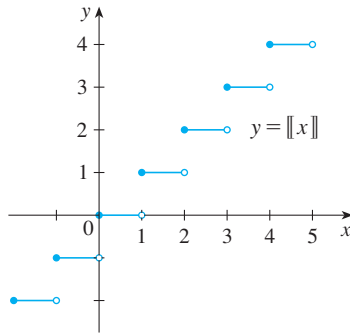


FIGURE 6
Greatest integer function

EXAMPLE 10 The **greatest integer function** is defined by $\llbracket x \rrbracket =$ the largest integer that is less than or equal to x . (For instance, $\llbracket 4 \rrbracket = 4$, $\llbracket 4.8 \rrbracket = 4$, $\llbracket \pi \rrbracket = 3$, $\llbracket \sqrt{2} \rrbracket = 1$, $\llbracket -\frac{1}{2} \rrbracket = -1$.) Show that $\lim_{x \rightarrow 3} \llbracket x \rrbracket$ does not exist.

SOLUTION The graph of the greatest integer function is shown in Figure 6. Since $\llbracket x \rrbracket = 3$ for $3 \leq x < 4$, we have

$$\lim_{x \rightarrow 3^+} \llbracket x \rrbracket = \lim_{x \rightarrow 3^+} 3 = 3$$

Since $\llbracket x \rrbracket = 2$ for $2 \leq x < 3$, we have

$$\lim_{x \rightarrow 3^-} \llbracket x \rrbracket = \lim_{x \rightarrow 3^-} 2 = 2$$

Because these one-sided limits are not equal, $\lim_{x \rightarrow 3} \llbracket x \rrbracket$ does not exist by Theorem 1.

The next two theorems give two additional properties of limits. Their proofs can be found in Appendix F.

2 Theorem If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a , then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$$

3 The Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L$$

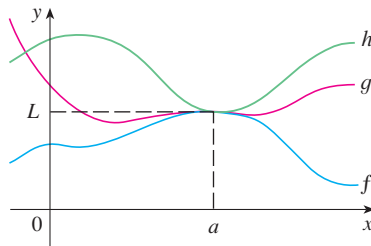


FIGURE 7

The Squeeze Theorem, which is sometimes called the Sandwich Theorem or the Pinching Theorem, is illustrated by Figure 7. It says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a , and if f and h have the same limit L at a , then g is forced to have the same limit L at a .

V EXAMPLE 11 Show that $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$.

SOLUTION First note that we **cannot** use

$$\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = \lim_{x \rightarrow 0} x^2 \cdot \lim_{x \rightarrow 0} \sin \frac{1}{x}$$

because $\lim_{x \rightarrow 0} \sin(1/x)$ does not exist (see Example 4 in Section 2.2).

Instead we apply the Squeeze Theorem, and so we need to find a function f smaller than $g(x) = x^2 \sin(1/x)$ and a function h bigger than g such that both $f(x)$ and $h(x)$

approach 0. To do this we use our knowledge of the sine function. Because the sine of any number lies between -1 and 1 , we can write

$$\boxed{4} \quad -1 \leq \sin \frac{1}{x} \leq 1$$

Any inequality remains true when multiplied by a positive number. We know that $x^2 \geq 0$ for all x and so, multiplying each side of the inequalities in $\boxed{4}$ by x^2 , we get

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$$

as illustrated by Figure 8. We know that

$$\lim_{x \rightarrow 0} x^2 = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (-x^2) = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$$

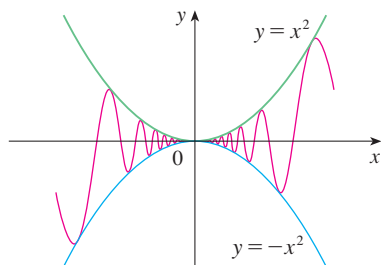


FIGURE 8
 $y = x^2 \sin(1/x)$

2.3 Exercises

1. Given that

$$\lim_{x \rightarrow 2} f(x) = 4 \quad \lim_{x \rightarrow 2} g(x) = -2 \quad \lim_{x \rightarrow 2} h(x) = 0$$

find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x \rightarrow 2} [f(x) + 5g(x)]$

(b) $\lim_{x \rightarrow 2} [g(x)]^3$

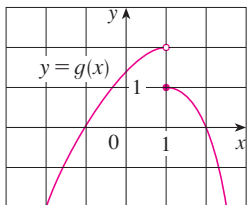
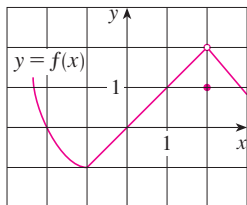
(c) $\lim_{x \rightarrow 2} \sqrt{f(x)}$

(d) $\lim_{x \rightarrow 2} \frac{3f(x)}{g(x)}$

(e) $\lim_{x \rightarrow 2} \frac{g(x)}{h(x)}$

(f) $\lim_{x \rightarrow 2} \frac{g(x)h(x)}{f(x)}$

2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.



(a) $\lim_{x \rightarrow 2} [f(x) + g(x)]$

(b) $\lim_{x \rightarrow 1} [f(x) + g(x)]$

(c) $\lim_{x \rightarrow 0} [f(x)g(x)]$

(d) $\lim_{x \rightarrow -1} \frac{f(x)}{g(x)}$

(e) $\lim_{x \rightarrow 2} [x^3 f(x)]$

(f) $\lim_{x \rightarrow 1} \sqrt{3 + f(x)}$

3–9 Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x \rightarrow 3} (5x^3 - 3x^2 + x - 6)$

4. $\lim_{x \rightarrow -1} (x^4 - 3x)(x^2 + 5x + 3)$

5. $\lim_{t \rightarrow -2} \frac{t^4 - 2}{2t^2 - 3t + 2}$

6. $\lim_{u \rightarrow -2} \sqrt{u^4 + 3u + 6}$

7. $\lim_{x \rightarrow 8} (1 + \sqrt[3]{x})(2 - 6x^2 + x^3)$

8. $\lim_{t \rightarrow 2} \left(\frac{t^2 - 2}{t^3 - 3t + 5} \right)^2$

9. $\lim_{x \rightarrow 2} \sqrt{\frac{2x^2 + 1}{3x - 2}}$

10. (a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \rightarrow 2} (x + 3)$$

is correct.



Graphing calculator or computer required

1. Homework Hints available at stewartcalculus.com

11–32 Evaluate the limit, if it exists.

11. $\lim_{x \rightarrow 5} \frac{x^2 - 6x + 5}{x - 5}$

12. $\lim_{x \rightarrow 4} \frac{x^2 - 4x}{x^2 - 3x - 4}$

13. $\lim_{x \rightarrow 5} \frac{x^2 - 5x + 6}{x - 5}$

14. $\lim_{x \rightarrow -1} \frac{x^2 - 4x}{x^2 - 3x - 4}$

15. $\lim_{t \rightarrow -3} \frac{t^2 - 9}{2t^2 + 7t + 3}$

16. $\lim_{x \rightarrow -1} \frac{2x^2 + 3x + 1}{x^2 - 2x - 3}$

17. $\lim_{h \rightarrow 0} \frac{(-5 + h)^2 - 25}{h}$

18. $\lim_{h \rightarrow 0} \frac{(2 + h)^3 - 8}{h}$

19. $\lim_{x \rightarrow -2} \frac{x + 2}{x^3 + 8}$

20. $\lim_{t \rightarrow 1} \frac{t^4 - 1}{t^3 - 1}$

21. $\lim_{h \rightarrow 0} \frac{\sqrt{9 + h} - 3}{h}$

22. $\lim_{u \rightarrow 2} \frac{\sqrt{4u + 1} - 3}{u - 2}$

23. $\lim_{x \rightarrow -4} \frac{\frac{1}{4} + \frac{1}{x}}{4 + x}$

24. $\lim_{x \rightarrow -1} \frac{x^2 + 2x + 1}{x^4 - 1}$

25. $\lim_{t \rightarrow 0} \frac{\sqrt{1 + t} - \sqrt{1 - t}}{t}$

26. $\lim_{t \rightarrow 0} \left(\frac{1}{t} - \frac{1}{t^2 + t} \right)$

27. $\lim_{x \rightarrow 16} \frac{4 - \sqrt{x}}{16x - x^2}$

28. $\lim_{h \rightarrow 0} \frac{(3 + h)^{-1} - 3^{-1}}{h}$

29. $\lim_{t \rightarrow 0} \left(\frac{1}{t\sqrt{1 + t}} - \frac{1}{t} \right)$

30. $\lim_{x \rightarrow -4} \frac{\sqrt{x^2 + 9} - 5}{x + 4}$

31. $\lim_{h \rightarrow 0} \frac{(x + h)^3 - x^3}{h}$

32. $\lim_{h \rightarrow 0} \frac{\frac{1}{(x + h)^2} - \frac{1}{x^2}}{h}$

 **33.** (a) Estimate the value of

$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{1 + 3x} - 1}$$

by graphing the function $f(x) = x/(\sqrt{1 + 3x} - 1)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.


 **34.** (a) Use a graph of

$$f(x) = \frac{\sqrt{3 + x} - \sqrt{3}}{x}$$

to estimate the value of $\lim_{x \rightarrow 0} f(x)$ to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Use the Limit Laws to find the exact value of the limit.

 **35.** Use the Squeeze Theorem to show that $\lim_{x \rightarrow 0} (x^2 \cos 20\pi x) = 0$. Illustrate by graphing the functions $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$, and $h(x) = x^2$ on the same screen.

 **36.** Use the Squeeze Theorem to show that

$$\lim_{x \rightarrow 0} \sqrt{x^3 + x^2} \sin \frac{\pi}{x} = 0$$

Illustrate by graphing the functions f , g , and h (in the notation of the Squeeze Theorem) on the same screen.

37. If $4x - 9 \leq f(x) \leq x^2 - 4x + 7$ for $x \geq 0$, find $\lim_{x \rightarrow 4} f(x)$.

38. If $2x \leq g(x) \leq x^4 - x^2 + 2$ for all x , evaluate $\lim_{x \rightarrow 1} g(x)$.

39. Prove that $\lim_{x \rightarrow 0} x^4 \cos \frac{2}{x} = 0$.

40. Prove that $\lim_{x \rightarrow 0^+} \sqrt{x} e^{\sin(\pi/x)} = 0$.

41–46 Find the limit, if it exists. If the limit does not exist, explain why.

41. $\lim_{x \rightarrow 3} (2x + |x - 3|)$

42. $\lim_{x \rightarrow -6} \frac{2x + 12}{|x + 6|}$

43. $\lim_{x \rightarrow 0.5^-} \frac{2x - 1}{|2x^3 - x^2|}$

44. $\lim_{x \rightarrow -2} \frac{2 - |x|}{2 + x}$

45. $\lim_{x \rightarrow 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

46. $\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

47. The *signum* (or *sign*) *function*, denoted by sgn , is defined by

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

(a) Sketch the graph of this function.

(b) Find each of the following limits or explain why it does not exist.

(i) $\lim_{x \rightarrow 0^+} \operatorname{sgn} x$

(ii) $\lim_{x \rightarrow 0^-} \operatorname{sgn} x$

(iii) $\lim_{x \rightarrow 0} \operatorname{sgn} x$

(iv) $\lim_{x \rightarrow 0} |\operatorname{sgn} x|$

48. Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x - 2)^2 & \text{if } x \geq 1 \end{cases}$$

(a) Find $\lim_{x \rightarrow 1^-} f(x)$ and $\lim_{x \rightarrow 1^+} f(x)$.

(b) Does $\lim_{x \rightarrow 1} f(x)$ exist?

(c) Sketch the graph of f .

49. Let $g(x) = \frac{x^2 + x - 6}{|x - 2|}$.

(a) Find

(i) $\lim_{x \rightarrow 2^+} g(x)$

(ii) $\lim_{x \rightarrow 2^-} g(x)$

(b) Does $\lim_{x \rightarrow 2} g(x)$ exist?

(c) Sketch the graph of g .

50. Let

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

(a) Evaluate each of the following, if it exists.

$$\begin{array}{lll} \text{(i)} \lim_{x \rightarrow 1^-} g(x) & \text{(ii)} \lim_{x \rightarrow 1} g(x) & \text{(iii)} g(1) \\ \text{(iv)} \lim_{x \rightarrow 2^-} g(x) & \text{(v)} \lim_{x \rightarrow 2^+} g(x) & \text{(vi)} \lim_{x \rightarrow 2} g(x) \end{array}$$

(b) Sketch the graph of g .51. (a) If the symbol $\llbracket \cdot \rrbracket$ denotes the greatest integer function defined in Example 10, evaluate

$$\text{(i)} \lim_{x \rightarrow -2^+} \llbracket x \rrbracket \quad \text{(ii)} \lim_{x \rightarrow -2} \llbracket x \rrbracket \quad \text{(iii)} \lim_{x \rightarrow -2.4} \llbracket x \rrbracket$$

(b) If n is an integer, evaluate

$$\text{(i)} \lim_{x \rightarrow n^-} \llbracket x \rrbracket \quad \text{(ii)} \lim_{x \rightarrow n^+} \llbracket x \rrbracket$$

(c) For what values of a does $\lim_{x \rightarrow a} \llbracket x \rrbracket$ exist?52. Let $f(x) = \llbracket \cos x \rrbracket$, $-\pi \leq x \leq \pi$.(a) Sketch the graph of f .

(b) Evaluate each limit, if it exists.

$$\begin{array}{ll} \text{(i)} \lim_{x \rightarrow 0} f(x) & \text{(ii)} \lim_{x \rightarrow (\pi/2)^-} f(x) \\ \text{(iii)} \lim_{x \rightarrow (\pi/2)^+} f(x) & \text{(iv)} \lim_{x \rightarrow \pi/2} f(x) \end{array}$$

(c) For what values of a does $\lim_{x \rightarrow a} f(x)$ exist?53. If $f(x) = \llbracket x \rrbracket + \llbracket -x \rrbracket$, show that $\lim_{x \rightarrow 2} f(x)$ exists but is not equal to $f(2)$.

54. In the theory of relativity, the Lorentz contraction formula

$$L = L_0 \sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v \rightarrow c^-} L$ and interpret the result. Why is a left-hand limit necessary?

55. If p is a polynomial, show that $\lim_{x \rightarrow a} p(x) = p(a)$.56. If r is a rational function, use Exercise 55 to show that $\lim_{x \rightarrow a} r(x) = r(a)$ for every number a in the domain of r .57. If $\lim_{x \rightarrow 1} \frac{f(x) - 8}{x - 1} = 10$, find $\lim_{x \rightarrow 1} f(x)$.58. If $\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 5$, find the following limits.

$$\text{(a)} \lim_{x \rightarrow 0} f(x) \quad \text{(b)} \lim_{x \rightarrow 0} \frac{f(x)}{x}$$

59. If

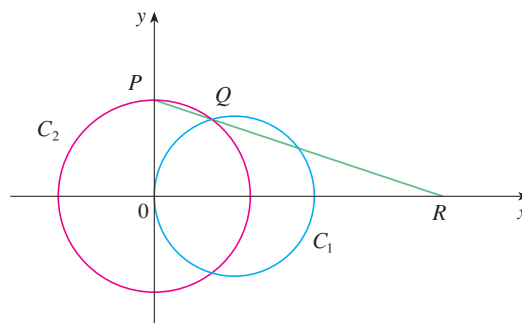
$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x \rightarrow 0} f(x) = 0$.60. Show by means of an example that $\lim_{x \rightarrow a} [f(x) + g(x)]$ may exist even though neither $\lim_{x \rightarrow a} f(x)$ nor $\lim_{x \rightarrow a} g(x)$ exists.61. Show by means of an example that $\lim_{x \rightarrow a} [f(x)g(x)]$ may exist even though neither $\lim_{x \rightarrow a} f(x)$ nor $\lim_{x \rightarrow a} g(x)$ exists.62. Evaluate $\lim_{x \rightarrow 2} \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1}$.63. Is there a number a such that

$$\lim_{x \rightarrow -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

64. The figure shows a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x -axis. What happens to R as C_2 shrinks, that is, as $r \rightarrow 0^+$?



2.4 The Precise Definition of a Limit

The intuitive definition of a limit given in Section 2.2 is inadequate for some purposes because such phrases as “ x is close to 2” and “ $f(x)$ gets closer and closer to L ” are vague. In order to be able to prove conclusively that

$$\lim_{x \rightarrow 0} \left(x^3 + \frac{\cos 5x}{10,000} \right) = 0.0001 \quad \text{or} \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

we must make the definition of a limit precise.

To motivate the precise definition of a limit, let's consider the function

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

Intuitively, it is clear that when x is close to 3 but $x \neq 3$, then $f(x)$ is close to 5, and so $\lim_{x \rightarrow 3} f(x) = 5$.

To obtain more detailed information about how $f(x)$ varies when x is close to 3, we ask the following question:

How close to 3 does x have to be so that $f(x)$ differs from 5 by less than 0.1?

The distance from x to 3 is $|x - 3|$ and the distance from $f(x)$ to 5 is $|f(x) - 5|$, so our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \quad \text{if} \quad |x - 3| < \delta \quad \text{but } x \neq 3$$

If $|x - 3| > 0$, then $x \neq 3$, so an equivalent formulation of our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \quad \text{if} \quad 0 < |x - 3| < \delta$$

Notice that if $0 < |x - 3| < (0.1)/2 = 0.05$, then

$$|f(x) - 5| = |(2x - 1) - 5| = |2x - 6| = 2|x - 3| < 2(0.05) = 0.1$$

that is, $|f(x) - 5| < 0.1$ if $0 < |x - 3| < 0.05$

Thus an answer to the problem is given by $\delta = 0.05$; that is, if x is within a distance of 0.05 from 3, then $f(x)$ will be within a distance of 0.1 from 5.

If we change the number 0.1 in our problem to the smaller number 0.01, then by using the same method we find that $f(x)$ will differ from 5 by less than 0.01 provided that x differs from 3 by less than $(0.01)/2 = 0.005$:

$$|f(x) - 5| < 0.01 \quad \text{if} \quad 0 < |x - 3| < 0.005$$

Similarly,

$$|f(x) - 5| < 0.001 \quad \text{if} \quad 0 < |x - 3| < 0.0005$$

The numbers 0.1, 0.01, and 0.001 that we have considered are *error tolerances* that we might allow. For 5 to be the precise limit of $f(x)$ as x approaches 3, we must not only be able to bring the difference between $f(x)$ and 5 below each of these three numbers; we must be able to bring it below *any* positive number. And, by the same reasoning, we can! If we write ε (the Greek letter epsilon) for an arbitrary positive number, then we find as before that

$$\boxed{1} \quad |f(x) - 5| < \varepsilon \quad \text{if} \quad 0 < |x - 3| < \delta = \frac{\varepsilon}{2}$$

This is a precise way of saying that $f(x)$ is close to 5 when x is close to 3 because $\boxed{1}$ says that we can make the values of $f(x)$ within an arbitrary distance ε from 5 by taking the values of x within a distance $\varepsilon/2$ from 3 (but $x \neq 3$).

It is traditional to use the Greek letter δ (delta) in this situation.

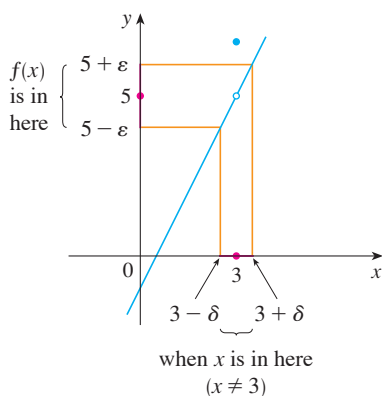


FIGURE 1

Note that [1] can be rewritten as follows:

$$\text{if } 3 - \delta < x < 3 + \delta \quad (x \neq 3) \quad \text{then} \quad 5 - \epsilon < f(x) < 5 + \epsilon$$

and this is illustrated in Figure 1. By taking the values of x ($\neq 3$) to lie in the interval $(3 - \delta, 3 + \delta)$ we can make the values of $f(x)$ lie in the interval $(5 - \epsilon, 5 + \epsilon)$.

Using [1] as a model, we give a precise definition of a limit.

2 Definition Let f be a function defined on some open interval that contains the number a , except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L** , and we write

$$\lim_{x \rightarrow a} f(x) = L$$

if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon$$

Since $|x - a|$ is the distance from x to a and $|f(x) - L|$ is the distance from $f(x)$ to L , and since ϵ can be arbitrarily small, the definition of a limit can be expressed in words as follows:

$\lim_{x \rightarrow a} f(x) = L$ means that the distance between $f(x)$ and L can be made arbitrarily small by taking the distance from x to a sufficiently small (but not 0).

Alternatively,

$\lim_{x \rightarrow a} f(x) = L$ means that the values of $f(x)$ can be made as close as we please to L by taking x close enough to a (but not equal to a).

We can also reformulate Definition 2 in terms of intervals by observing that the inequality $|x - a| < \delta$ is equivalent to $-\delta < x - a < \delta$, which in turn can be written as $a - \delta < x < a + \delta$. Also $0 < |x - a|$ is true if and only if $x - a \neq 0$, that is, $x \neq a$. Similarly, the inequality $|f(x) - L| < \epsilon$ is equivalent to the pair of inequalities $L - \epsilon < f(x) < L + \epsilon$. Therefore, in terms of intervals, Definition 2 can be stated as follows:

$\lim_{x \rightarrow a} f(x) = L$ means that for every $\epsilon > 0$ (no matter how small ϵ is) we can find $\delta > 0$ such that if x lies in the open interval $(a - \delta, a + \delta)$ and $x \neq a$, then $f(x)$ lies in the open interval $(L - \epsilon, L + \epsilon)$.

We interpret this statement geometrically by representing a function by an arrow diagram as in Figure 2, where f maps a subset of \mathbb{R} onto another subset of \mathbb{R} .

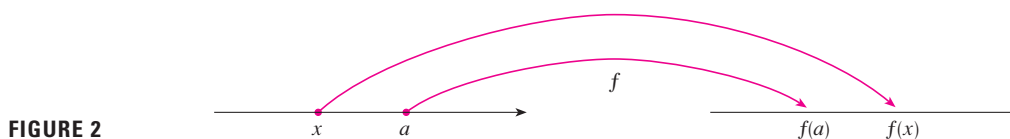
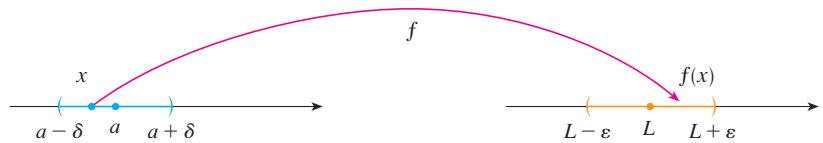


FIGURE 2

The definition of limit says that if any small interval $(L - \epsilon, L + \epsilon)$ is given around L , then we can find an interval $(a - \delta, a + \delta)$ around a such that f maps all the points in $(a - \delta, a + \delta)$ (except possibly a) into the interval $(L - \epsilon, L + \epsilon)$. (See Figure 3.)

FIGURE 3



Another geometric interpretation of limits can be given in terms of the graph of a function. If $\varepsilon > 0$ is given, then we draw the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ and the graph of f . (See Figure 4.) If $\lim_{x \rightarrow a} f(x) = L$, then we can find a number $\delta > 0$ such that if we restrict x to lie in the interval $(a - \delta, a + \delta)$ and take $x \neq a$, then the curve $y = f(x)$ lies between the lines $y = L - \varepsilon$ and $y = L + \varepsilon$. (See Figure 5.) You can see that if such a δ has been found, then any smaller δ will also work.

It is important to realize that the process illustrated in Figures 4 and 5 must work for *every* positive number ε , no matter how small it is chosen. Figure 6 shows that if a smaller ε is chosen, then a smaller δ may be required.

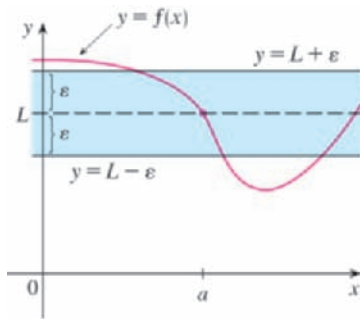


FIGURE 4

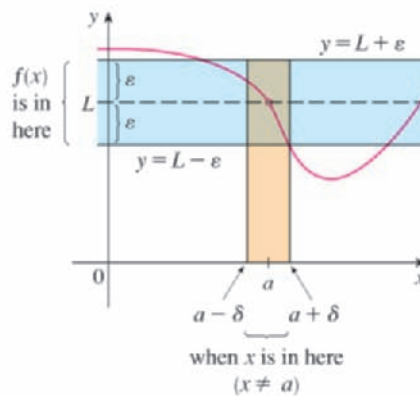


FIGURE 5

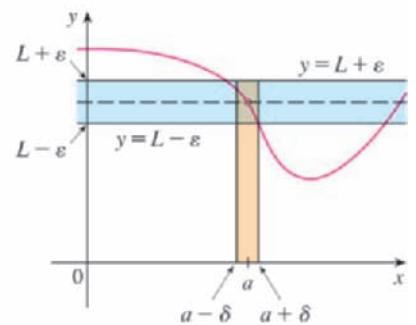


FIGURE 6

EXAMPLE 1 Use a graph to find a number δ such that

$$\text{if } |x - 1| < \delta \quad \text{then} \quad |(x^3 - 5x + 6) - 2| < 0.2$$

In other words, find a number δ that corresponds to $\varepsilon = 0.2$ in the definition of a limit for the function $f(x) = x^3 - 5x + 6$ with $a = 1$ and $L = 2$.

SOLUTION A graph of f is shown in Figure 7; we are interested in the region near the point $(1, 2)$. Notice that we can rewrite the inequality

$$|(x^3 - 5x + 6) - 2| < 0.2$$

as

$$1.8 < x^3 - 5x + 6 < 2.2$$

So we need to determine the values of x for which the curve $y = x^3 - 5x + 6$ lies between the horizontal lines $y = 1.8$ and $y = 2.2$. Therefore we graph the curves $y = x^3 - 5x + 6$, $y = 1.8$, and $y = 2.2$ near the point $(1, 2)$ in Figure 8. Then we use the cursor to estimate that the x -coordinate of the point of intersection of the line $y = 2.2$ and the curve $y = x^3 - 5x + 6$ is about 0.911. Similarly, $y = x^3 - 5x + 6$ intersects the line $y = 1.8$ when $x \approx 1.124$. So, rounding to be safe, we can say that

$$\text{if } 0.92 < x < 1.12 \quad \text{then} \quad 1.8 < x^3 - 5x + 6 < 2.2$$

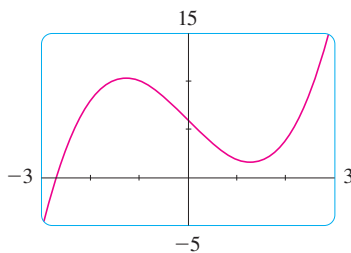


FIGURE 7

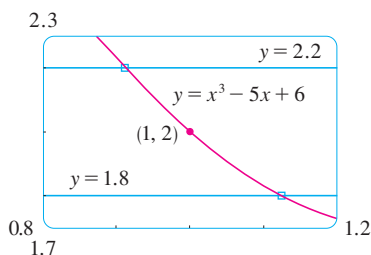


FIGURE 8

This interval $(0.92, 1.12)$ is not symmetric about $x = 1$. The distance from $x = 1$ to the left endpoint is $1 - 0.92 = 0.08$ and the distance to the right endpoint is 0.12 . We can choose δ to be the smaller of these numbers, that is, $\delta = 0.08$. Then we can rewrite our inequalities in terms of distances as follows:

$$\text{if } |x - 1| < 0.08 \quad \text{then} \quad |(x^3 - 5x + 6) - 2| < 0.2$$

This just says that by keeping x within 0.08 of 1 , we are able to keep $f(x)$ within 0.2 of 2 .

Although we chose $\delta = 0.08$, any smaller positive value of δ would also have worked.

The graphical procedure in Example 1 gives an illustration of the definition for $\varepsilon = 0.2$, but it does not *prove* that the limit is equal to 2 . A proof has to provide a δ for *every* ε .

In proving limit statements it may be helpful to think of the definition of limit as a challenge. First it challenges you with a number ε . Then you must be able to produce a suitable δ . You have to be able to do this for *every* $\varepsilon > 0$, not just a particular ε .

Imagine a contest between two people, A and B, and imagine yourself to be B. Person A stipulates that the fixed number L should be approximated by the values of $f(x)$ to within a degree of accuracy ε (say, 0.01). Person B then responds by finding a number δ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. Then A may become more exacting and challenge B with a smaller value of ε (say, 0.0001). Again B has to respond by finding a corresponding δ . Usually the smaller the value of ε , the smaller the corresponding value of δ must be. If B always wins, no matter how small A makes ε , then $\lim_{x \rightarrow a} f(x) = L$.

TEC In Module 2.4/2.6 you can explore the precise definition of a limit both graphically and numerically.

V EXAMPLE 2 Prove that $\lim_{x \rightarrow 3} (4x - 5) = 7$.

SOLUTION

1. Preliminary analysis of the problem (guessing a value for δ). Let ε be a given positive number. We want to find a number δ such that

$$\text{if } 0 < |x - 3| < \delta \quad \text{then} \quad |(4x - 5) - 7| < \varepsilon$$

But $|(4x - 5) - 7| = |4x - 12| = |4(x - 3)| = 4|x - 3|$. Therefore we want δ such that

$$\text{if } 0 < |x - 3| < \delta \quad \text{then} \quad 4|x - 3| < \varepsilon$$

that is, $\text{if } 0 < |x - 3| < \delta \quad \text{then} \quad |x - 3| < \frac{\varepsilon}{4}$

This suggests that we should choose $\delta = \varepsilon/4$.

2. Proof (showing that this δ works). Given $\varepsilon > 0$, choose $\delta = \varepsilon/4$. If $0 < |x - 3| < \delta$, then

$$|(4x - 5) - 7| = |4x - 12| = 4|x - 3| < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon$$

Thus

$$\text{if } 0 < |x - 3| < \delta \quad \text{then} \quad |(4x - 5) - 7| < \varepsilon$$

Therefore, by the definition of a limit,

$$\lim_{x \rightarrow 3} (4x - 5) = 7$$

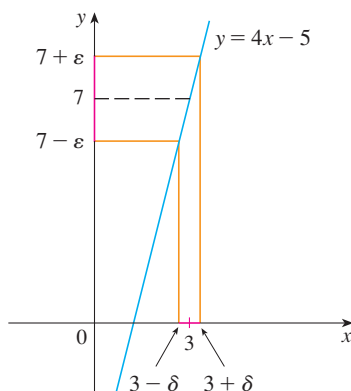


FIGURE 9

This example is illustrated by Figure 9.

Note that in the solution of Example 2 there were two stages—guessing and proving. We made a preliminary analysis that enabled us to guess a value for δ . But then in the second stage we had to go back and prove in a careful, logical fashion that we had made a correct guess. This procedure is typical of much of mathematics. Sometimes it is necessary to first make an intelligent guess about the answer to a problem and then prove that the guess is correct.

The intuitive definitions of one-sided limits that were given in Section 2.2 can be precisely reformulated as follows.

Cauchy and Limits

After the invention of calculus in the 17th century, there followed a period of free development of the subject in the 18th century. Mathematicians like the Bernoulli brothers and Euler were eager to exploit the power of calculus and boldly explored the consequences of this new and wonderful mathematical theory without worrying too much about whether their proofs were completely correct.

The 19th century, by contrast, was the Age of Rigor in mathematics. There was a movement to go back to the foundations of the subject—to provide careful definitions and rigorous proofs. At the forefront of this movement was the French mathematician Augustin-Louis Cauchy (1789–1857), who started out as a military engineer before becoming a mathematics professor in Paris. Cauchy took Newton's idea of a limit, which was kept alive in the 18th century by the French mathematician Jean d'Alembert, and made it more precise. His definition of a limit reads as follows: "When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the *limit* of all the others." But when Cauchy used this definition in examples and proofs, he often employed delta-epsilon inequalities similar to the ones in this section. A typical Cauchy proof starts with: "Designate by δ and ε two very small numbers; . . ." He used ε because of the correspondence between epsilon and the French word *erreur* and δ because delta corresponds to *différence*. Later, the German mathematician Karl Weierstrass (1815–1897) stated the definition of a limit exactly as in our Definition 2.

3 Definition of Left-Hand Limit

$$\lim_{x \rightarrow a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } a - \delta < x < a \quad \text{then} \quad |f(x) - L| < \varepsilon$$

4 Definition of Right-Hand Limit

$$\lim_{x \rightarrow a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } a < x < a + \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

Notice that Definition 3 is the same as Definition 2 except that x is restricted to lie in the *left* half $(a - \delta, a)$ of the interval $(a - \delta, a + \delta)$. In Definition 4, x is restricted to lie in the *right* half $(a, a + \delta)$ of the interval $(a - \delta, a + \delta)$.

V EXAMPLE 3 Use Definition 4 to prove that $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$.

SOLUTION

1. *Guessing a value for δ .* Let ε be a given positive number. Here $a = 0$ and $L = 0$, so we want to find a number δ such that

$$\text{if } 0 < x < \delta \quad \text{then} \quad |\sqrt{x} - 0| < \varepsilon$$

that is, $\text{if } 0 < x < \delta \quad \text{then} \quad \sqrt{x} < \varepsilon$

or, squaring both sides of the inequality $\sqrt{x} < \varepsilon$, we get

$$\text{if } 0 < x < \delta \quad \text{then} \quad x < \varepsilon^2$$

This suggests that we should choose $\delta = \varepsilon^2$.

2. *Showing that this δ works.* Given $\varepsilon > 0$, let $\delta = \varepsilon^2$. If $0 < x < \delta$, then

$$\sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$$

so $|\sqrt{x} - 0| < \varepsilon$

According to Definition 4, this shows that $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$.

EXAMPLE 4 Prove that $\lim_{x \rightarrow 3} x^2 = 9$.

SOLUTION

1. *Guessing a value for δ .* Let $\varepsilon > 0$ be given. We have to find a number $\delta > 0$ such that

$$\text{if } 0 < |x - 3| < \delta \quad \text{then} \quad |x^2 - 9| < \varepsilon$$

To connect $|x^2 - 9|$ with $|x - 3|$ we write $|x^2 - 9| = |(x + 3)(x - 3)|$. Then we want

$$\text{if } 0 < |x - 3| < \delta \quad \text{then} \quad |x + 3||x - 3| < \varepsilon$$

Notice that if we can find a positive constant C such that $|x + 3| < C$, then

$$|x + 3||x - 3| < C|x - 3|$$

and we can make $C|x - 3| < \varepsilon$ by taking $|x - 3| < \varepsilon/C = \delta$.

We can find such a number C if we restrict x to lie in some interval centered at 3. In fact, since we are interested only in values of x that are close to 3, it is reasonable to assume that x is within a distance 1 from 3, that is, $|x - 3| < 1$. Then $2 < x < 4$, so $5 < x + 3 < 7$. Thus we have $|x + 3| < 7$, and so $C = 7$ is a suitable choice for the constant.

But now there are two restrictions on $|x - 3|$, namely

$$|x - 3| < 1 \quad \text{and} \quad |x - 3| < \frac{\varepsilon}{C} = \frac{\varepsilon}{7}$$

To make sure that both of these inequalities are satisfied, we take δ to be the smaller of the two numbers 1 and $\varepsilon/7$. The notation for this is $\delta = \min\{1, \varepsilon/7\}$.

2. *Showing that this δ works.* Given $\varepsilon > 0$, let $\delta = \min\{1, \varepsilon/7\}$. If $0 < |x - 3| < \delta$, then $|x - 3| < 1 \Rightarrow 2 < x < 4 \Rightarrow |x + 3| < 7$ (as in part 1). We also have $|x - 3| < \varepsilon/7$, so

$$|x^2 - 9| = |x + 3||x - 3| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon$$

This shows that $\lim_{x \rightarrow 3} x^2 = 9$.

As Example 4 shows, it is not always easy to prove that limit statements are true using the ε, δ definition. In fact, if we had been given a more complicated function such as $f(x) = (6x^2 - 8x + 9)/(2x^2 - 1)$, a proof would require a great deal of ingenuity. Fortunately this is unnecessary because the Limit Laws stated in Section 2.3 can be proved using Definition 2, and then the limits of complicated functions can be found rigorously from the Limit Laws without resorting to the definition directly.

For instance, we prove the Sum Law: If $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$ both exist, then

$$\lim_{x \rightarrow a} [f(x) + g(x)] = L + M$$

The remaining laws are proved in the exercises and in Appendix F.

PROOF OF THE SUM LAW Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad |f(x) + g(x) - (L + M)| < \varepsilon$$

Triangle Inequality:

$$|a + b| \leq |a| + |b|$$

(See Appendix A.)

Using the Triangle Inequality we can write

$$\begin{aligned} \boxed{5} \quad |f(x) + g(x) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \end{aligned}$$

We make $|f(x) + g(x) - (L + M)|$ less than ε by making each of the terms $|f(x) - L|$ and $|g(x) - M|$ less than $\varepsilon/2$.

Since $\varepsilon/2 > 0$ and $\lim_{x \rightarrow a} f(x) = L$, there exists a number $\delta_1 > 0$ such that

$$\text{if } 0 < |x - a| < \delta_1 \quad \text{then} \quad |f(x) - L| < \frac{\varepsilon}{2}$$

Similarly, since $\lim_{x \rightarrow a} g(x) = M$, there exists a number $\delta_2 > 0$ such that

$$\text{if } 0 < |x - a| < \delta_2 \quad \text{then} \quad |g(x) - M| < \frac{\varepsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$, the smaller of the numbers δ_1 and δ_2 . Notice that

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad 0 < |x - a| < \delta_1 \quad \text{and} \quad 0 < |x - a| < \delta_2$$

$$\text{and so} \quad |f(x) - L| < \frac{\varepsilon}{2} \quad \text{and} \quad |g(x) - M| < \frac{\varepsilon}{2}$$

Therefore, by $\boxed{5}$,

$$\begin{aligned} |f(x) + g(x) - (L + M)| &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

To summarize,

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad |f(x) + g(x) - (L + M)| < \varepsilon$$

Thus, by the definition of a limit,

$$\lim_{x \rightarrow a} [f(x) + g(x)] = L + M$$

Infinite Limits

Infinite limits can also be defined in a precise way. The following is a precise version of Definition 4 in Section 2.2.

6 Definition Let f be a function defined on some open interval that contains the number a , except possibly at a itself. Then

$$\lim_{x \rightarrow a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad f(x) > M$$

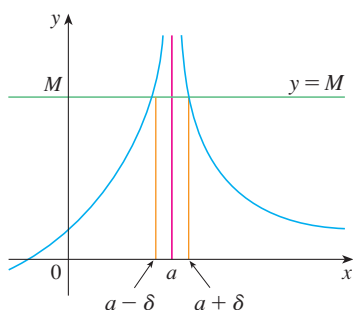


FIGURE 10

This says that the values of $f(x)$ can be made arbitrarily large (larger than any given number M) by taking x close enough to a (within a distance δ , where δ depends on M , but with $x \neq a$). A geometric illustration is shown in Figure 10.

Given any horizontal line $y = M$, we can find a number $\delta > 0$ such that if we restrict x to lie in the interval $(a - \delta, a + \delta)$ but $x \neq a$, then the curve $y = f(x)$ lies above the line $y = M$. You can see that if a larger M is chosen, then a smaller δ may be required.

V EXAMPLE 5 Use Definition 6 to prove that $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$.

SOLUTION Let M be a given positive number. We want to find a number δ such that

$$\text{if } 0 < |x| < \delta \quad \text{then} \quad 1/x^2 > M$$

$$\text{But} \quad \frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff |x| < \frac{1}{\sqrt{M}}$$

So if we choose $\delta = 1/\sqrt{M}$ and $0 < |x| < \delta = 1/\sqrt{M}$, then $1/x^2 > M$. This shows that $1/x^2 \rightarrow \infty$ as $x \rightarrow 0$.

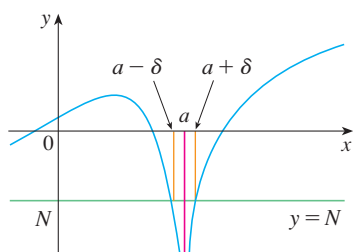


FIGURE 11

Similarly, the following is a precise version of Definition 5 in Section 2.2. It is illustrated by Figure 11.

7 Definition Let f be a function defined on some open interval that contains the number a , except possibly at a itself. Then

$$\lim_{x \rightarrow a} f(x) = -\infty$$

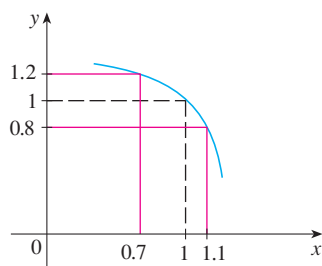
means that for every negative number N there is a positive number δ such that

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad f(x) < N$$

2.4 Exercises

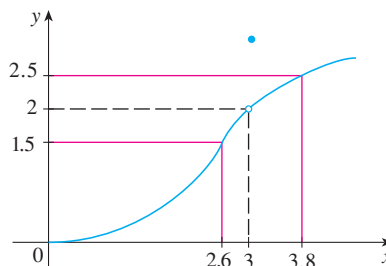
1. Use the given graph of f to find a number δ such that

$$\text{if } |x - 1| < \delta \quad \text{then} \quad |f(x) - 1| < 0.2$$



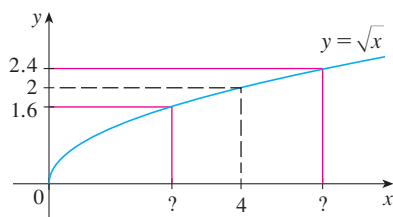
2. Use the given graph of f to find a number δ such that

$$\text{if } 0 < |x - 3| < \delta \quad \text{then} \quad |f(x) - 2| < 0.5$$



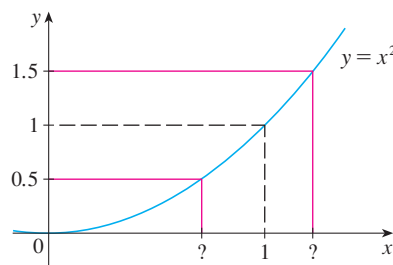
3. Use the given graph of $f(x) = \sqrt{x}$ to find a number δ such that

$$\text{if } |x - 4| < \delta \quad \text{then} \quad |\sqrt{x} - 2| < 0.4$$



4. Use the given graph of $f(x) = x^2$ to find a number δ such that

$$\text{if } |x - 1| < \delta \quad \text{then} \quad |x^2 - 1| < \frac{1}{2}$$



5. Use a graph to find a number δ such that

$$\text{if } \left| x - \frac{\pi}{4} \right| < \delta \quad \text{then} \quad \left| \tan x - 1 \right| < 0.2$$

6. Use a graph to find a number δ such that

$$\text{if } |x - 1| < \delta \quad \text{then} \quad \left| \frac{2x}{x^2 + 4} - 0.4 \right| < 0.1$$

7. For the limit

$$\lim_{x \rightarrow 2} (x^3 - 3x + 4) = 6$$

illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.2$ and $\varepsilon = 0.1$.

8. For the limit

$$\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x} = 2$$

illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

9. Given that $\lim_{x \rightarrow \pi/2} \tan^2 x = \infty$, illustrate Definition 6 by finding values of δ that correspond to (a) $M = 1000$ and (b) $M = 10,000$.

10. Use a graph to find a number δ such that

$$\text{if } 5 < x < 5 + \delta \quad \text{then} \quad \frac{x^2}{\sqrt{x} - 5} > 100$$

11. A machinist is required to manufacture a circular metal disk with area 1000 cm^2 .
- What radius produces such a disk?
 - If the machinist is allowed an error tolerance of $\pm 5 \text{ cm}^2$ in the area of the disk, how close to the ideal radius in part (a) must the machinist control the radius?
 - In terms of the ε, δ definition of $\lim_{x \rightarrow a} f(x) = L$, what is x ? What is $f(x)$? What is a ? What is L ? What value of ε is given? What is the corresponding value of δ ?



12. A crystal growth furnace is used in research to determine how best to manufacture crystals used in electronic components for the space shuttle. For proper growth of the crystal, the temperature must be controlled accurately by adjusting the input power. Suppose the relationship is given by

$$T(w) = 0.1w^2 + 2.155w + 20$$

where T is the temperature in degrees Celsius and w is the power input in watts.

- How much power is needed to maintain the temperature at 200°C ?
- If the temperature is allowed to vary from 200°C by up to $\pm 1^\circ\text{C}$, what range of wattage is allowed for the input power?
- In terms of the ε, δ definition of $\lim_{x \rightarrow a} f(x) = L$, what is x ? What is $f(x)$? What is a ? What is L ? What value of ε is given? What is the corresponding value of δ ?

13. (a) Find a number δ such that if $|x - 2| < \delta$, then $|4x - 8| < \varepsilon$, where $\varepsilon = 0.1$.
(b) Repeat part (a) with $\varepsilon = 0.01$.

14. Given that $\lim_{x \rightarrow 2} (5x - 7) = 3$, illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.1$, $\varepsilon = 0.05$, and $\varepsilon = 0.01$.

15–18 Prove the statement using the ε, δ definition of a limit and illustrate with a diagram like Figure 9.

15. $\lim_{x \rightarrow 3} (1 + \frac{1}{3}x) = 2$

16. $\lim_{x \rightarrow 4} (2x - 5) = 3$

17. $\lim_{x \rightarrow -3} (1 - 4x) = 13$

18. $\lim_{x \rightarrow -2} (3x + 5) = -1$

19–32 Prove the statement using the ε, δ definition of a limit.

19. $\lim_{x \rightarrow 1} \frac{2 + 4x}{3} = 2$

20. $\lim_{x \rightarrow 10} (3 - \frac{4}{5}x) = -5$

21. $\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2} = 5$

22. $\lim_{x \rightarrow -1.5} \frac{9 - 4x^2}{3 + 2x} = 6$

23. $\lim_{x \rightarrow a} x = a$

24. $\lim_{x \rightarrow a} c = c$

25. $\lim_{x \rightarrow 0} x^2 = 0$

26. $\lim_{x \rightarrow 0} x^3 = 0$

27. $\lim_{x \rightarrow 0} |x| = 0$

28. $\lim_{x \rightarrow -6^+} \sqrt[8]{6 + x} = 0$

29. $\lim_{x \rightarrow 2} (x^2 - 4x + 5) = 1$

30. $\lim_{x \rightarrow 2} (x^2 + 2x - 7) = 1$

31. $\lim_{x \rightarrow -2} (x^2 - 1) = 3$

32. $\lim_{x \rightarrow 2} x^3 = 8$

33. Verify that another possible choice of δ for showing that $\lim_{x \rightarrow 3} x^2 = 9$ in Example 4 is $\delta = \min\{2, \varepsilon/8\}$.34. Verify, by a geometric argument, that the largest possible choice of δ for showing that $\lim_{x \rightarrow 3} x^2 = 9$ is $\delta = \sqrt{9 + \varepsilon} - 3$.

- CAS** 35. (a) For the limit $\lim_{x \rightarrow 1} (x^3 + x + 1) = 3$, use a graph to find a value of δ that corresponds to $\varepsilon = 0.4$.
 (b) By using a computer algebra system to solve the cubic equation $x^3 + x + 1 = 3 + \varepsilon$, find the largest possible value of δ that works for any given $\varepsilon > 0$.
 (c) Put $\varepsilon = 0.4$ in your answer to part (b) and compare with your answer to part (a).

36. Prove that $\lim_{x \rightarrow 2} \frac{1}{x} = \frac{1}{2}$.

37. Prove that $\lim_{x \rightarrow a} \sqrt{x} = \sqrt{a}$ if $a > 0$.

$$\left[\text{Hint: Use } |\sqrt{x} - \sqrt{a}| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}}. \right]$$

38. If H is the Heaviside function defined in Example 6 in Section 2.2, prove, using Definition 2, that $\lim_{t \rightarrow 0} H(t)$ does not exist. [Hint: Use an indirect proof as follows. Suppose thatthe limit is L . Take $\varepsilon = \frac{1}{2}$ in the definition of a limit and try to arrive at a contradiction.]39. If the function f is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x \rightarrow 0} f(x)$ does not exist.

40. By comparing Definitions 2, 3, and 4, prove Theorem 1 in Section 2.3.

41. How close to -3 do we have to take x so that

$$\frac{1}{(x + 3)^4} > 10,000$$

42. Prove, using Definition 6, that $\lim_{x \rightarrow -3} \frac{1}{(x + 3)^4} = \infty$.43. Prove that $\lim_{x \rightarrow 0^+} \ln x = -\infty$.44. Suppose that $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = c$, where c is a real number. Prove each statement.

(a) $\lim_{x \rightarrow a} [f(x) + g(x)] = \infty$

(b) $\lim_{x \rightarrow a} [f(x)g(x)] = \infty$ if $c > 0$

(c) $\lim_{x \rightarrow a} [f(x)g(x)] = -\infty$ if $c < 0$

2.5 Continuity

We noticed in Section 2.3 that the limit of a function as x approaches a can often be found simply by calculating the value of the function at a . Functions with this property are called *continuous at a* . We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language. (A continuous process is one that takes place gradually, without interruption or abrupt change.)

As illustrated in Figure 1, if f is continuous, then the points $(x, f(x))$ on the graph of f approach the point $(a, f(a))$ on the graph. So there is no gap in the curve.

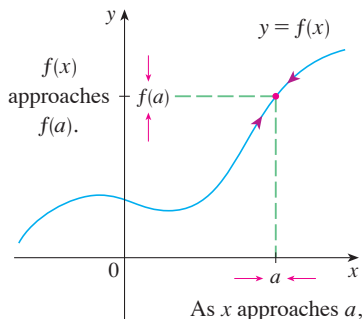


FIGURE 1

1 Definition A function f is **continuous at a number a** if

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Notice that Definition 1 implicitly requires three things if f is continuous at a :

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \rightarrow a} f(x)$ exists
3. $\lim_{x \rightarrow a} f(x) = f(a)$

The definition says that f is continuous at a if $f(x)$ approaches $f(a)$ as x approaches a . Thus a continuous function f has the property that a small change in x produces only a

small change in $f(x)$. In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small.

If f is defined near a (in other words, f is defined on an open interval containing a , except perhaps at a), we say that f is **discontinuous at a** (or f has a **discontinuity at a**) if f is not continuous at a .

Physical phenomena are usually continuous. For instance, the displacement or velocity of a vehicle varies continuously with time, as does a person's height. But discontinuities do occur in such situations as electric currents. [See Example 6 in Section 2.2, where the Heaviside function is discontinuous at 0 because $\lim_{t \rightarrow 0} H(t)$ does not exist.]

Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it. The graph can be drawn without removing your pen from the paper.

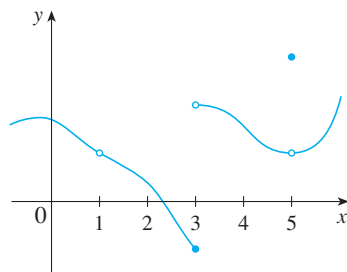


FIGURE 2

EXAMPLE 1 Figure 2 shows the graph of a function f . At which numbers is f discontinuous? Why?

SOLUTION It looks as if there is a discontinuity when $a = 1$ because the graph has a break there. The official reason that f is discontinuous at 1 is that $f(1)$ is not defined.

The graph also has a break when $a = 3$, but the reason for the discontinuity is different. Here, $f(3)$ is defined, but $\lim_{x \rightarrow 3} f(x)$ does not exist (because the left and right limits are different). So f is discontinuous at 3.

What about $a = 5$? Here, $f(5)$ is defined and $\lim_{x \rightarrow 5} f(x)$ exists (because the left and right limits are the same). But

$$\lim_{x \rightarrow 5} f(x) \neq f(5)$$

So f is discontinuous at 5.

Now let's see how to detect discontinuities when a function is defined by a formula.

EXAMPLE 2 Where are each of the following functions discontinuous?

$$\begin{aligned} \text{(a) } f(x) &= \frac{x^2 - x - 2}{x - 2} & \text{(b) } f(x) &= \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \\ \text{(c) } f(x) &= \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases} & \text{(d) } f(x) &= \llbracket x \rrbracket \end{aligned}$$

SOLUTION

(a) Notice that $f(2)$ is not defined, so f is discontinuous at 2. Later we'll see why f is continuous at all other numbers.

(b) Here $f(0) = 1$ is defined but

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{1}{x^2}$$

does not exist. (See Example 8 in Section 2.2.) So f is discontinuous at 0.

(c) Here $f(2) = 1$ is defined and

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 1)}{x - 2} = \lim_{x \rightarrow 2} (x + 1) = 3$$

exists. But

$$\lim_{x \rightarrow 2} f(x) \neq f(2)$$

so f is not continuous at 2.

(d) The greatest integer function $f(x) = \llbracket x \rrbracket$ has discontinuities at all of the integers because $\lim_{x \rightarrow n} \llbracket x \rrbracket$ does not exist if n is an integer. (See Example 10 and Exercise 51 in Section 2.3.)

Figure 3 shows the graphs of the functions in Example 2. In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph. The kind of discontinuity illustrated in parts (a) and (c) is called **removable** because we could remove the discontinuity by redefining f at just the single number 2. [The function $g(x) = x + 1$ is continuous.] The discontinuity in part (b) is called an **infinite discontinuity**. The discontinuities in part (d) are called **jump discontinuities** because the function “jumps” from one value to another.

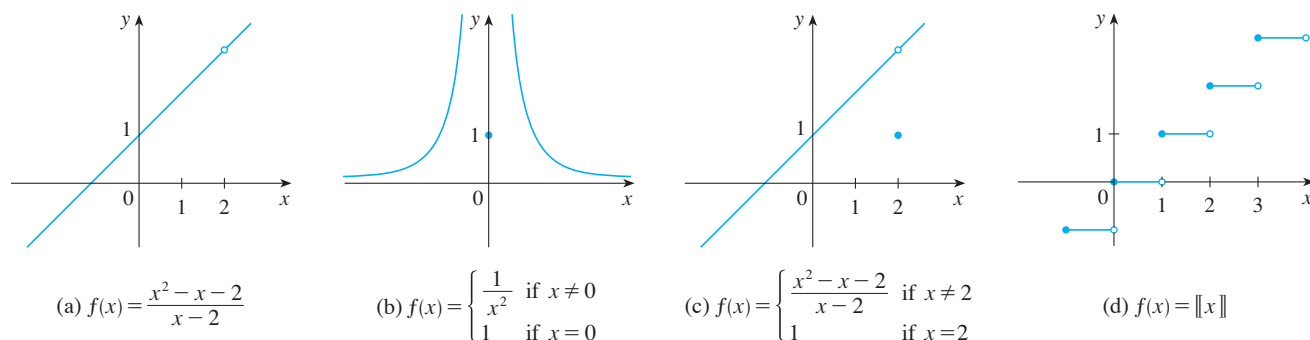


FIGURE 3
Graphs of the functions in Example 2

2 Definition A function f is **continuous from the right at a number a** if

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

and f is **continuous from the left at a** if

$$\lim_{x \rightarrow a^-} f(x) = f(a)$$

EXAMPLE 3 At each integer n , the function $f(x) = \llbracket x \rrbracket$ [see Figure 3(d)] is continuous from the right but discontinuous from the left because

$$\lim_{x \rightarrow n^+} f(x) = \lim_{x \rightarrow n^+} \llbracket x \rrbracket = n = f(n)$$

but

$$\lim_{x \rightarrow n^-} f(x) = \lim_{x \rightarrow n^-} \llbracket x \rrbracket = n - 1 \neq f(n)$$

3 Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

EXAMPLE 4 Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

SOLUTION If $-1 < a < 1$, then using the Limit Laws, we have

$$\begin{aligned}
 \lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} (1 - \sqrt{1 - x^2}) \\
 &= 1 - \lim_{x \rightarrow a} \sqrt{1 - x^2} && \text{(by Laws 2 and 7)} \\
 &= 1 - \sqrt{\lim_{x \rightarrow a} (1 - x^2)} && \text{(by 11)} \\
 &= 1 - \sqrt{1 - a^2} && \text{(by 2, 7, and 9)} \\
 &= f(a)
 \end{aligned}$$

Thus, by Definition 1, f is continuous at a if $-1 < a < 1$. Similar calculations show that

$$\lim_{x \rightarrow -1^+} f(x) = 1 = f(-1) \quad \text{and} \quad \lim_{x \rightarrow 1^-} f(x) = 1 = f(1)$$

so f is continuous from the right at -1 and continuous from the left at 1 . Therefore, according to Definition 3, f is continuous on $[-1, 1]$.

The graph of f is sketched in Figure 4. It is the lower half of the circle

$$x^2 + (y - 1)^2 = 1$$

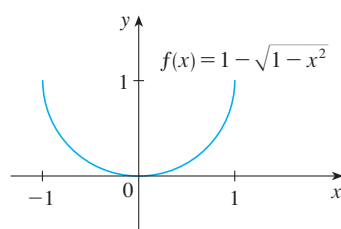


FIGURE 4

Instead of always using Definitions 1, 2, and 3 to verify the continuity of a function as we did in Example 4, it is often convenient to use the next theorem, which shows how to build up complicated continuous functions from simple ones.

4 Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a :

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\frac{f}{g}$ if $g(a) \neq 0$

PROOF Each of the five parts of this theorem follows from the corresponding Limit Law in Section 2.3. For instance, we give the proof of part 1. Since f and g are continuous at a , we have

$$\lim_{x \rightarrow a} f(x) = f(a) \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = g(a)$$

Therefore

$$\begin{aligned}
 \lim_{x \rightarrow a} (f + g)(x) &= \lim_{x \rightarrow a} [f(x) + g(x)] \\
 &= \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) && \text{(by Law 1)} \\
 &= f(a) + g(a) \\
 &= (f + g)(a)
 \end{aligned}$$

This shows that $f + g$ is continuous at a .

It follows from Theorem 4 and Definition 3 that if f and g are continuous on an interval, then so are the functions $f + g$, $f - g$, cf , fg , and (if g is never 0) f/g . The following theorem was stated in Section 2.3 as the Direct Substitution Property.

5 Theorem

- (a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.
- (b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

PROOF

(a) A polynomial is a function of the form

$$P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

where c_0, c_1, \dots, c_n are constants. We know that

$$\lim_{x \rightarrow a} c_0 = c_0 \quad (\text{by Law 7})$$

$$\text{and} \quad \lim_{x \rightarrow a} x^m = a^m \quad m = 1, 2, \dots, n \quad (\text{by 9})$$

This equation is precisely the statement that the function $f(x) = x^m$ is a continuous function. Thus, by part 3 of Theorem 4, the function $g(x) = cx^m$ is continuous. Since P is a sum of functions of this form and a constant function, it follows from part 1 of Theorem 4 that P is continuous.

(b) A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain of f is $D = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$. We know from part (a) that P and Q are continuous everywhere. Thus, by part 5 of Theorem 4, f is continuous at every number in D . ■

As an illustration of Theorem 5, observe that the volume of a sphere varies continuously with its radius because the formula $V(r) = \frac{4}{3}\pi r^3$ shows that V is a polynomial function of r . Likewise, if a ball is thrown vertically into the air with a velocity of 50 ft/s, then the height of the ball in feet t seconds later is given by the formula $h = 50t - 16t^2$. Again this is a polynomial function, so the height is a continuous function of the elapsed time.

Knowledge of which functions are continuous enables us to evaluate some limits very quickly, as the following example shows. Compare it with Example 2(b) in Section 2.3.

EXAMPLE 5 Find $\lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$.

SOLUTION The function

$$f(x) = \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

is rational, so by Theorem 5 it is continuous on its domain, which is $\{x \mid x \neq \frac{5}{3}\}$.

Therefore

$$\begin{aligned}\lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} &= \lim_{x \rightarrow -2} f(x) = f(-2) \\ &= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11}\end{aligned}$$

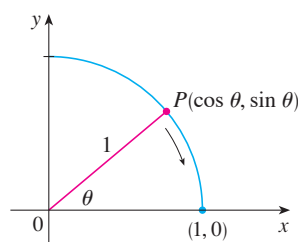


FIGURE 5

Another way to establish the limits in [6] is to use the Squeeze Theorem with the inequality $\sin \theta < \theta$ (for $\theta > 0$), which is proved in Section 3.3.

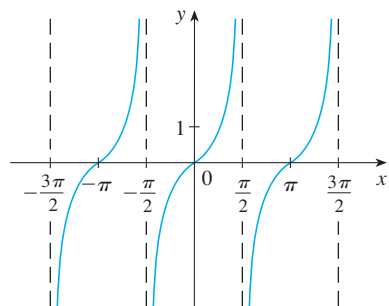


FIGURE 6 $y = \tan x$

The inverse trigonometric functions are reviewed in Section 1.6.

It turns out that most of the familiar functions are continuous at every number in their domains. For instance, Limit Law 10 (page 100) is exactly the statement that root functions are continuous.

From the appearance of the graphs of the sine and cosine functions (Figure 18 in Section 1.2), we would certainly guess that they are continuous. We know from the definitions of $\sin \theta$ and $\cos \theta$ that the coordinates of the point P in Figure 5 are $(\cos \theta, \sin \theta)$. As $\theta \rightarrow 0$, we see that P approaches the point $(1, 0)$ and so $\cos \theta \rightarrow 1$ and $\sin \theta \rightarrow 0$. Thus

$$\boxed{6} \quad \lim_{\theta \rightarrow 0} \cos \theta = 1 \quad \lim_{\theta \rightarrow 0} \sin \theta = 0$$

Since $\cos 0 = 1$ and $\sin 0 = 0$, the equations in [6] assert that the cosine and sine functions are continuous at 0. The addition formulas for cosine and sine can then be used to deduce that these functions are continuous everywhere (see Exercises 60 and 61).

It follows from part 5 of Theorem 4 that

$$\tan x = \frac{\sin x}{\cos x}$$

is continuous except where $\cos x = 0$. This happens when x is an odd integer multiple of $\pi/2$, so $y = \tan x$ has infinite discontinuities when $x = \pm\pi/2, \pm3\pi/2, \pm5\pi/2$, and so on (see Figure 6).

The inverse function of any continuous one-to-one function is also continuous. (This fact is proved in Appendix F, but our geometric intuition makes it seem plausible: The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$. So if the graph of f has no break in it, neither does the graph of f^{-1} .) Thus the inverse trigonometric functions are continuous.

In Section 1.5 we defined the exponential function $y = a^x$ so as to fill in the holes in the graph of $y = a^x$ where x is rational. In other words, the very definition of $y = a^x$ makes it a continuous function on \mathbb{R} . Therefore its inverse function $y = \log_a x$ is continuous on $(0, \infty)$.

7 Theorem The following types of functions are continuous at every number in their domains:

polynomials	rational functions	root functions
trigonometric functions	inverse trigonometric functions	
exponential functions	logarithmic functions	

EXAMPLE 6 Where is the function $f(x) = \frac{\ln x + \tan^{-1}x}{x^2 - 1}$ continuous?

SOLUTION We know from Theorem 7 that the function $y = \ln x$ is continuous for $x > 0$ and $y = \tan^{-1}x$ is continuous on \mathbb{R} . Thus, by part 1 of Theorem 4, $y = \ln x + \tan^{-1}x$ is continuous on $(0, \infty)$. The denominator, $y = x^2 - 1$, is a polynomial, so it is continuous

everywhere. Therefore, by part 5 of Theorem 4, f is continuous at all positive numbers x except where $x^2 - 1 = 0$. So f is continuous on the intervals $(0, 1)$ and $(1, \infty)$.

EXAMPLE 7 Evaluate $\lim_{x \rightarrow \pi} \frac{\sin x}{2 + \cos x}$.

SOLUTION Theorem 7 tells us that $y = \sin x$ is continuous. The function in the denominator, $y = 2 + \cos x$, is the sum of two continuous functions and is therefore continuous. Notice that this function is never 0 because $\cos x \geq -1$ for all x and so $2 + \cos x > 0$ everywhere. Thus the ratio

$$f(x) = \frac{\sin x}{2 + \cos x}$$

is continuous everywhere. Hence, by the definition of a continuous function,

$$\lim_{x \rightarrow \pi} \frac{\sin x}{2 + \cos x} = \lim_{x \rightarrow \pi} f(x) = f(\pi) = \frac{\sin \pi}{2 + \cos \pi} = \frac{0}{2 - 1} = 0$$

Another way of combining continuous functions f and g to get a new continuous function is to form the composite function $f \circ g$. This fact is a consequence of the following theorem.

This theorem says that a limit symbol can be moved through a function symbol if the function is continuous and the limit exists. In other words, the order of these two symbols can be reversed.

8 Theorem If f is continuous at b and $\lim_{x \rightarrow a} g(x) = b$, then $\lim_{x \rightarrow a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right)$$

Intuitively, Theorem 8 is reasonable because if x is close to a , then $g(x)$ is close to b , and since f is continuous at b , if $g(x)$ is close to b , then $f(g(x))$ is close to $f(b)$. A proof of Theorem 8 is given in Appendix F.

EXAMPLE 8 Evaluate $\lim_{x \rightarrow 1} \arcsin\left(\frac{1 - \sqrt{x}}{1 - x}\right)$.

SOLUTION Because \arcsin is a continuous function, we can apply Theorem 8:

$$\begin{aligned} \lim_{x \rightarrow 1} \arcsin\left(\frac{1 - \sqrt{x}}{1 - x}\right) &= \arcsin\left(\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x}\right) \\ &= \arcsin\left(\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{(1 - \sqrt{x})(1 + \sqrt{x})}\right) \\ &= \arcsin\left(\lim_{x \rightarrow 1} \frac{1}{1 + \sqrt{x}}\right) \\ &= \arcsin \frac{1}{2} = \frac{\pi}{6} \end{aligned}$$

Let's now apply Theorem 8 in the special case where $f(x) = \sqrt[n]{x}$, with n being a positive integer. Then

$$f(g(x)) = \sqrt[n]{g(x)}$$

and

$$f\left(\lim_{x \rightarrow a} g(x)\right) = \sqrt[n]{\lim_{x \rightarrow a} g(x)}$$

If we put these expressions into Theorem 8, we get

$$\lim_{x \rightarrow a} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x \rightarrow a} g(x)}$$

and so Limit Law 11 has now been proved. (We assume that the roots exist.)

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a .

This theorem is often expressed informally by saying “a continuous function of a continuous function is a continuous function.”

PROOF Since g is continuous at a , we have

$$\lim_{x \rightarrow a} g(x) = g(a)$$

Since f is continuous at $b = g(a)$, we can apply Theorem 8 to obtain

$$\lim_{x \rightarrow a} f(g(x)) = f(g(a))$$

which is precisely the statement that the function $h(x) = f(g(x))$ is continuous at a ; that is, $f \circ g$ is continuous at a .

V EXAMPLE 9 Where are the following functions continuous?

- (a) $h(x) = \sin(x^2)$ (b) $F(x) = \ln(1 + \cos x)$

SOLUTION

(a) We have $h(x) = f(g(x))$, where

$$g(x) = x^2 \quad \text{and} \quad f(x) = \sin x$$

Now g is continuous on \mathbb{R} since it is a polynomial, and f is also continuous everywhere. Thus $h = f \circ g$ is continuous on \mathbb{R} by Theorem 9.

(b) We know from Theorem 7 that $f(x) = \ln x$ is continuous and $g(x) = 1 + \cos x$ is continuous (because both $y = 1$ and $y = \cos x$ are continuous). Therefore, by Theorem 9, $F(x) = f(g(x))$ is continuous wherever it is defined. Now $\ln(1 + \cos x)$ is defined when $1 + \cos x > 0$. So it is undefined when $\cos x = -1$, and this happens when $x = \pm\pi, \pm3\pi, \dots$. Thus F has discontinuities when x is an odd multiple of π and is continuous on the intervals between these values (see Figure 7).

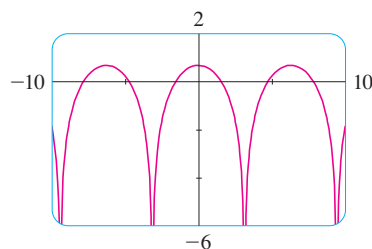


FIGURE 7
 $y = \ln(1 + \cos x)$

An important property of continuous functions is expressed by the following theorem, whose proof is found in more advanced books on calculus.

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. It is illustrated by Figure 8. Note that the value N can be taken on once [as in part (a)] or more than once [as in part (b)].

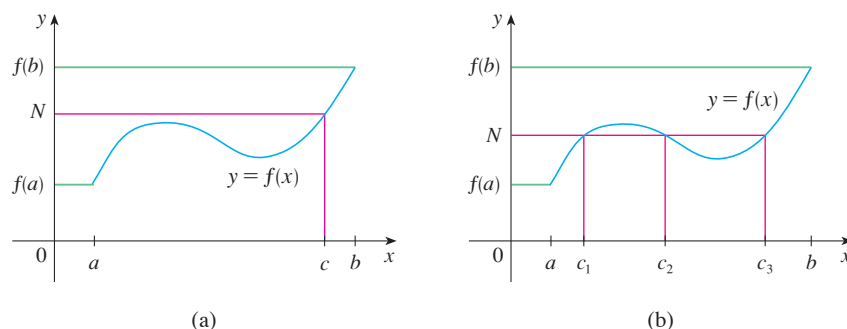


FIGURE 8

If we think of a continuous function as a function whose graph has no hole or break, then it is easy to believe that the Intermediate Value Theorem is true. In geometric terms it says that if any horizontal line $y = N$ is given between $y = f(a)$ and $y = f(b)$ as in Figure 9, then the graph of f can't jump over the line. It must intersect $y = N$ somewhere.

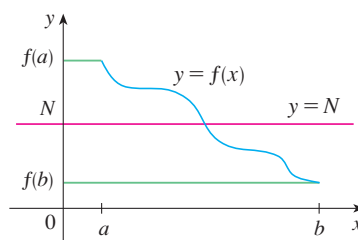


FIGURE 9

It is important that the function f in Theorem 10 be continuous. The Intermediate Value Theorem is not true in general for discontinuous functions (see Exercise 48).

One use of the Intermediate Value Theorem is in locating roots of equations as in the following example.

V EXAMPLE 10 Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

SOLUTION Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. We are looking for a solution of the given equation, that is, a number c between 1 and 2 such that $f(c) = 0$. Therefore we take $a = 1$, $b = 2$, and $N = 0$ in Theorem 10. We have

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$

and

$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Thus $f(1) < 0 < f(2)$; that is, $N = 0$ is a number between $f(1)$ and $f(2)$. Now f is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number c between 1 and 2 such that $f(c) = 0$. In other words, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one root c in the interval $(1, 2)$.

In fact, we can locate a root more precisely by using the Intermediate Value Theorem again. Since

$$f(1.2) = -0.128 < 0 \quad \text{and} \quad f(1.3) = 0.548 > 0$$

a root must lie between 1.2 and 1.3. A calculator gives, by trial and error,

$$f(1.22) = -0.007008 < 0 \quad \text{and} \quad f(1.23) = 0.056068 > 0$$

so a root lies in the interval $(1.22, 1.23)$.

We can use a graphing calculator or computer to illustrate the use of the Intermediate Value Theorem in Example 10. Figure 10 shows the graph of f in the viewing rectangle $[-1, 3]$ by $[-3, 3]$ and you can see that the graph crosses the x -axis between 1 and 2. Figure 11 shows the result of zooming in to the viewing rectangle $[1.2, 1.3]$ by $[-0.2, 0.2]$.

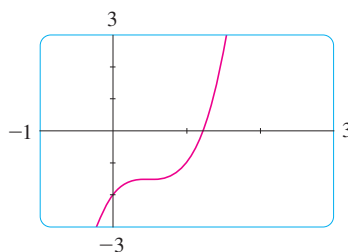


FIGURE 10

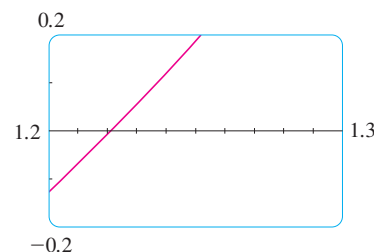
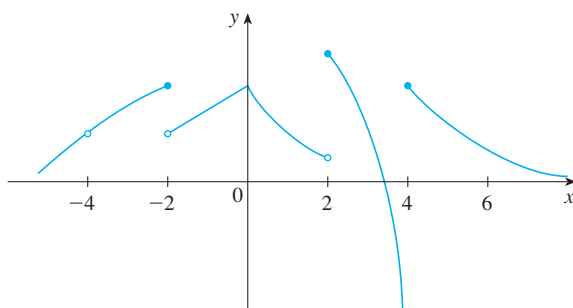


FIGURE 11

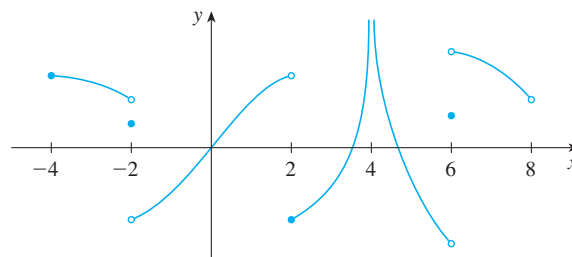
In fact, the Intermediate Value Theorem plays a role in the very way these graphing devices work. A computer calculates a finite number of points on the graph and turns on the pixels that contain these calculated points. It assumes that the function is continuous and takes on all the intermediate values between two consecutive points. The computer therefore connects the pixels by turning on the intermediate pixels.

2.5 Exercises

- Write an equation that expresses the fact that a function f is continuous at the number 4.
- If f is continuous on $(-\infty, \infty)$, what can you say about its graph?
- From the graph of f , state the numbers at which f is discontinuous and explain why.
 - For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left, or neither.



- From the graph of g , state the intervals on which g is continuous.



5–8 Sketch the graph of a function f that is continuous except for the stated discontinuity.

- Discontinuous, but continuous from the right, at 2
- Discontinuities at -1 and 4 , but continuous from the left at -1 and from the right at 4
- Removable discontinuity at 3, jump discontinuity at 5
- Neither left nor right continuous at -2 , continuous only from the left at 2

9. The toll T charged for driving on a certain stretch of a toll road is \$5 except during rush hours (between 7 AM and 10 AM and between 4 PM and 7 PM) when the toll is \$7.

- Sketch a graph of T as a function of the time t , measured in hours past midnight.
 - Discuss the discontinuities of this function and their significance to someone who uses the road.
10. Explain why each function is continuous or discontinuous.
- The temperature at a specific location as a function of time
 - The temperature at a specific time as a function of the distance due west from New York City
 - The altitude above sea level as a function of the distance due west from New York City
 - The cost of a taxi ride as a function of the distance traveled
 - The current in the circuit for the lights in a room as a function of time

11. Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim_{x \rightarrow 2} [3f(x) + f(x)g(x)] = 36$. Find $f(2)$.

12–14 Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a .

12. $f(x) = 3x^4 - 5x + \sqrt[3]{x^2 + 4}$, $a = 2$

13. $f(x) = (x + 2x^3)^4$, $a = -1$

14. $h(t) = \frac{2t - 3t^2}{1 + t^3}$, $a = 1$

15–16 Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

15. $f(x) = \frac{2x + 3}{x - 2}$, $(2, \infty)$

16. $g(x) = 2\sqrt{3 - x}$, $(-\infty, 3]$

17–22 Explain why the function is discontinuous at the given number a . Sketch the graph of the function.

17. $f(x) = \frac{1}{x + 2}$, $a = -2$

18. $f(x) = \begin{cases} \frac{1}{x + 2} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$, $a = -2$

19. $f(x) = \begin{cases} e^x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$, $a = 0$

20. $f(x) = \begin{cases} \frac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$, $a = 1$

21. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}$, $a = 0$

22. $f(x) = \begin{cases} \frac{2x^2 - 5x - 3}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$, $a = 3$

23–24 How would you “remove the discontinuity” of f ? In other words, how would you define $f(2)$ in order to make f continuous at 2?

23. $f(x) = \frac{x^2 - x - 2}{x - 2}$

24. $f(x) = \frac{x^3 - 8}{x^2 - 4}$

25–32 Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain. State the domain.

25. $F(x) = \frac{2x^2 - x - 1}{x^2 + 1}$

26. $G(x) = \frac{x^2 + 1}{2x^2 - x - 1}$

27. $Q(x) = \frac{\sqrt[3]{x - 2}}{x^3 - 2}$


28. $R(t) = \frac{e^{\sin t}}{2 + \cos \pi t}$

29. $A(t) = \arcsin(1 + 2t)$

30. $B(x) = \frac{\tan x}{\sqrt{4 - x^2}}$

31. $M(x) = \sqrt{1 + \frac{1}{x}}$

32. $N(r) = \tan^{-1}(1 + e^{-r^2})$

 33–34 Locate the discontinuities of the function and illustrate by graphing.

33. $y = \frac{1}{1 + e^{1/x}}$

34. $y = \ln(\tan^2 x)$

35–38 Use continuity to evaluate the limit.

35. $\lim_{x \rightarrow 4} \frac{5 + \sqrt{x}}{\sqrt{5 + x}}$

36. $\lim_{x \rightarrow \pi} \sin(x + \sin x)$

37. $\lim_{x \rightarrow 1} e^{x^2 - x}$

38. $\lim_{x \rightarrow 2} \arctan\left(\frac{x^2 - 4}{3x^2 - 6x}\right)$

39–40 Show that f is continuous on $(-\infty, \infty)$.

39. $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$

40. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$

41–43 Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f .

41. $f(x) = \begin{cases} 1 + x^2 & \text{if } x \leq 0 \\ 2 - x & \text{if } 0 < x \leq 2 \\ (x - 2)^2 & \text{if } x > 2 \end{cases}$

$$42. f(x) = \begin{cases} x + 1 & \text{if } x \leq 1 \\ 1/x & \text{if } 1 < x < 3 \\ \sqrt{x-3} & \text{if } x \geq 3 \end{cases}$$

$$43. f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$

44. The gravitational force exerted by the planet Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \frac{GMr}{R^3} & \text{if } r < R \\ \frac{GM}{r^2} & \text{if } r \geq R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r ?

45. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

46. Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leq x < 3 \\ 2x - a + b & \text{if } x \geq 3 \end{cases}$$

47. Which of the following functions f has a removable discontinuity at a ? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous at a .

$$(a) f(x) = \frac{x^4 - 1}{x - 1}, \quad a = 1$$

$$(b) f(x) = \frac{x^3 - x^2 - 2x}{x - 2}, \quad a = 2$$

$$(c) f(x) = \llbracket \sin x \rrbracket, \quad a = \pi$$

48. Suppose that a function f is continuous on $[0, 1]$ except at 0.25 and that $f(0) = 1$ and $f(1) = 3$. Let $N = 2$. Sketch two possible graphs of f , one showing that f might not satisfy the conclusion of the Intermediate Value Theorem and one showing that f might still satisfy the conclusion of the Intermediate Value Theorem (even though it doesn't satisfy the hypothesis).

49. If $f(x) = x^2 + 10 \sin x$, show that there is a number c such that $f(c) = 1000$.

50. Suppose f is continuous on $[1, 5]$ and the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) = 8$, explain why $f(3) > 6$.

51–54 Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

51. $x^4 + x - 3 = 0$, $(1, 2)$

52. $\sqrt[3]{x} = 1 - x$, $(0, 1)$

53. $e^x = 3 - 2x$, $(0, 1)$

54. $\sin x = x^2 - x$, $(1, 2)$

55–56 (a) Prove that the equation has at least one real root.
(b) Use your calculator to find an interval of length 0.01 that contains a root.

55. $\cos x = x^3$

56. $\ln x = 3 - 2x$



57–58 (a) Prove that the equation has at least one real root.
(b) Use your graphing device to find the root correct to three decimal places.

57. $100e^{-x/100} = 0.01x^2$

58. $\arctan x = 1 - x$

59. Prove that f is continuous at a if and only if

$$\lim_{h \rightarrow 0} f(a + h) = f(a)$$

60. To prove that sine is continuous, we need to show that $\lim_{x \rightarrow a} \sin x = \sin a$ for every real number a . By Exercise 59 an equivalent statement is that

$$\lim_{h \rightarrow 0} \sin(a + h) = \sin a$$

Use [6] to show that this is true.

61. Prove that cosine is a continuous function.

62. (a) Prove Theorem 4, part 3.
(b) Prove Theorem 4, part 5.

63. For what values of x is f continuous?

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

64. For what values of x is g continuous?

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$$

65. Is there a number that is exactly 1 more than its cube?

66. If a and b are positive numbers, prove that the equation

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0$$

has at least one solution in the interval $(-1, 1)$.

67. Show that the function

$$f(x) = \begin{cases} x^4 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous on $(-\infty, \infty)$.

68. (a) Show that the absolute value function $F(x) = |x|$ is continuous everywhere.

(b) Prove that if f is a continuous function on an interval, then so is $|f|$.

(c) Is the converse of the statement in part (b) also true? In other words, if $|f|$ is continuous, does it follow that f is continuous? If so, prove it. If not, find a counterexample.

69. A Tibetan monk leaves the monastery at 7:00 AM and takes his usual path to the top of the mountain, arriving at 7:00 PM. The following morning, he starts at 7:00 AM at the top and takes the same path back, arriving at the monastery at 7:00 PM. Use the Intermediate Value Theorem to show that there is a point on the path that the monk will cross at exactly the same time of day on both days.

2.6 Limits at Infinity; Horizontal Asymptotes

x	$f(x)$
0	-1
± 1	0
± 2	0.600000
± 3	0.800000
± 4	0.882353
± 5	0.923077
± 10	0.980198
± 50	0.999200
± 100	0.999800
± 1000	0.999998

In Sections 2.2 and 2.4 we investigated infinite limits and vertical asymptotes. There we let x approach a number and the result was that the values of y became arbitrarily large (positive or negative). In this section we let x become arbitrarily large (positive or negative) and see what happens to y .

Let's begin by investigating the behavior of the function f defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as x becomes large. The table at the left gives values of this function correct to six decimal places, and the graph of f has been drawn by a computer in Figure 1.

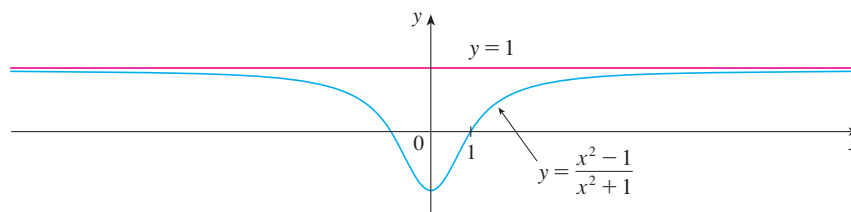


FIGURE 1

As x grows larger and larger you can see that the values of $f(x)$ get closer and closer to 1. In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large. This situation is expressed symbolically by writing

$$\lim_{x \rightarrow \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

In general, we use the notation

$$\lim_{x \rightarrow \infty} f(x) = L$$

to indicate that the values of $f(x)$ approach L as x becomes larger and larger.

1 Definition Let f be a function defined on some interval (a, ∞) . Then

$$\lim_{x \rightarrow \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

Another notation for $\lim_{x \rightarrow \infty} f(x) = L$ is

$$f(x) \rightarrow L \quad \text{as} \quad x \rightarrow \infty$$

The symbol ∞ does not represent a number. Nonetheless, the expression $\lim_{x \rightarrow \infty} f(x) = L$ is often read as

“the limit of $f(x)$, as x approaches infinity, is L ”

or

“the limit of $f(x)$, as x becomes infinite, is L ”

or

“the limit of $f(x)$, as x increases without bound, is L ”

The meaning of such phrases is given by Definition 1. A more precise definition, similar to the ε, δ definition of Section 2.4, is given at the end of this section.

Geometric illustrations of Definition 1 are shown in Figure 2. Notice that there are many ways for the graph of f to approach the line $y = L$ (which is called a *horizontal asymptote*) as we look to the far right of each graph.

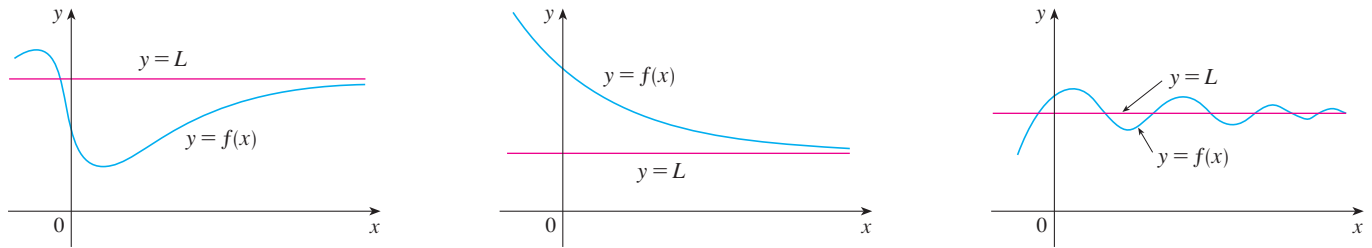


FIGURE 2

Examples illustrating $\lim_{x \rightarrow \infty} f(x) = L$

Referring back to Figure 1, we see that for numerically large negative values of x , the values of $f(x)$ are close to 1. By letting x decrease through negative values without bound, we can make $f(x)$ as close to 1 as we like. This is expressed by writing

$$\lim_{x \rightarrow -\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

The general definition is as follows.

2 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \rightarrow -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

Again, the symbol $-\infty$ does not represent a number, but the expression $\lim_{x \rightarrow -\infty} f(x) = L$ is often read as

“the limit of $f(x)$, as x approaches negative infinity, is L ”

Definition 2 is illustrated in Figure 3. Notice that the graph approaches the line $y = L$ as we look to the far left of each graph.

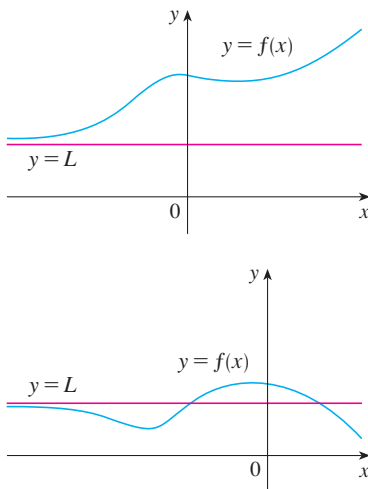


FIGURE 3

Examples illustrating $\lim_{x \rightarrow -\infty} f(x) = L$

3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = L$$

For instance, the curve illustrated in Figure 1 has the line $y = 1$ as a horizontal asymptote because

$$\lim_{x \rightarrow \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

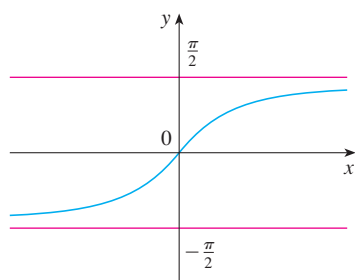


FIGURE 4
 $y = \tan^{-1}x$

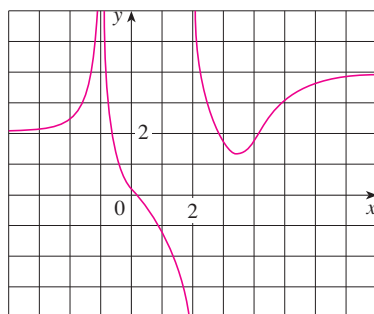


FIGURE 5

An example of a curve with two horizontal asymptotes is $y = \tan^{-1}x$. (See Figure 4.) In fact,

4

$$\lim_{x \rightarrow -\infty} \tan^{-1}x = -\frac{\pi}{2} \quad \lim_{x \rightarrow \infty} \tan^{-1}x = \frac{\pi}{2}$$

so both of the lines $y = -\pi/2$ and $y = \pi/2$ are horizontal asymptotes. (This follows from the fact that the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of \tan .)

EXAMPLE 1 Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown in Figure 5.

SOLUTION We see that the values of $f(x)$ become large as $x \rightarrow -1$ from both sides, so

$$\lim_{x \rightarrow -1} f(x) = \infty$$

Notice that $f(x)$ becomes large negative as x approaches 2 from the left, but large positive as x approaches 2 from the right. So

$$\lim_{x \rightarrow 2^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow 2^+} f(x) = \infty$$

Thus both of the lines $x = -1$ and $x = 2$ are vertical asymptotes.

As x becomes large, it appears that $f(x)$ approaches 4. But as x decreases through negative values, $f(x)$ approaches 2. So

$$\lim_{x \rightarrow \infty} f(x) = 4 \quad \text{and} \quad \lim_{x \rightarrow -\infty} f(x) = 2$$

This means that both $y = 4$ and $y = 2$ are horizontal asymptotes.

EXAMPLE 2 Find $\lim_{x \rightarrow \infty} \frac{1}{x}$ and $\lim_{x \rightarrow -\infty} \frac{1}{x}$.

SOLUTION Observe that when x is large, $1/x$ is small. For instance,

$$\frac{1}{100} = 0.01 \quad \frac{1}{10,000} = 0.0001 \quad \frac{1}{1,000,000} = 0.000001$$

In fact, by taking x large enough, we can make $1/x$ as close to 0 as we please. Therefore, according to Definition 1, we have

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

Similar reasoning shows that when x is large negative, $1/x$ is small negative, so we also have

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

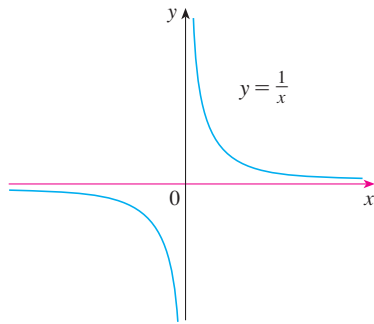


FIGURE 6

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0, \quad \lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

It follows that the line $y = 0$ (the x -axis) is a horizontal asymptote of the curve $y = 1/x$. (This is an equilateral hyperbola; see Figure 6.)

Most of the Limit Laws that were given in Section 2.3 also hold for limits at infinity. It can be proved that the *Limit Laws listed in Section 2.3 (with the exception of Laws 9 and 10) are also valid if “ $x \rightarrow a$ ” is replaced by “ $x \rightarrow \infty$ ” or “ $x \rightarrow -\infty$.”* In particular, if we combine Laws 6 and 11 with the results of Example 2, we obtain the following important rule for calculating limits.

5 Theorem If $r > 0$ is a rational number, then

$$\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x , then

$$\lim_{x \rightarrow -\infty} \frac{1}{x^r} = 0$$

V EXAMPLE 3 Evaluate

$$\lim_{x \rightarrow \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

and indicate which properties of limits are used at each stage.

SOLUTION As x becomes large, both numerator and denominator become large, so it isn't obvious what happens to their ratio. We need to do some preliminary algebra.

To evaluate the limit at infinity of any rational function, we first divide both the numerator and denominator by the highest power of x that occurs in the denominator. (We may assume that $x \neq 0$, since we are interested only in large values of x .) In this case the highest power of x in the denominator is x^2 , so we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} &= \lim_{x \rightarrow \infty} \frac{\frac{3x^2 - x - 2}{x^2}}{\frac{5x^2 + 4x + 1}{x^2}} = \lim_{x \rightarrow \infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}} \\ &= \frac{\lim_{x \rightarrow \infty} \left(3 - \frac{1}{x} - \frac{2}{x^2} \right)}{\lim_{x \rightarrow \infty} \left(5 + \frac{4}{x} + \frac{1}{x^2} \right)} && \text{(by Limit Law 5)} \\ &= \frac{\lim_{x \rightarrow \infty} 3 - \lim_{x \rightarrow \infty} \frac{1}{x} - 2 \lim_{x \rightarrow \infty} \frac{1}{x^2}}{\lim_{x \rightarrow \infty} 5 + 4 \lim_{x \rightarrow \infty} \frac{1}{x} + \lim_{x \rightarrow \infty} \frac{1}{x^2}} && \text{(by 1, 2, and 3)} \\ &= \frac{3 - 0 - 0}{5 + 0 + 0} && \text{(by 7 and Theorem 5)} \\ &= \frac{3}{5} \end{aligned}$$

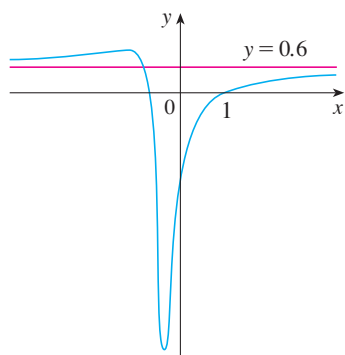


FIGURE 7

$$y = \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

A similar calculation shows that the limit as $x \rightarrow -\infty$ is also $\frac{3}{5}$. Figure 7 illustrates the results of these calculations by showing how the graph of the given rational function approaches the horizontal asymptote $y = \frac{3}{5}$.

EXAMPLE 4 Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

SOLUTION Dividing both numerator and denominator by x and using the properties of limits, we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} &= \lim_{x \rightarrow \infty} \frac{\sqrt{2 + \frac{1}{x^2}}}{3 - \frac{5}{x}} \quad (\text{since } \sqrt{x^2} = x \text{ for } x > 0) \\ &= \frac{\lim_{x \rightarrow \infty} \sqrt{2 + \frac{1}{x^2}}}{\lim_{x \rightarrow \infty} \left(3 - \frac{5}{x}\right)} = \frac{\sqrt{\lim_{x \rightarrow \infty} 2 + \lim_{x \rightarrow \infty} \frac{1}{x^2}}}{\lim_{x \rightarrow \infty} 3 - 5 \lim_{x \rightarrow \infty} \frac{1}{x}} = \frac{\sqrt{2 + 0}}{3 - 5 \cdot 0} = \frac{\sqrt{2}}{3} \end{aligned}$$

Therefore the line $y = \sqrt{2}/3$ is a horizontal asymptote of the graph of f .

In computing the limit as $x \rightarrow -\infty$, we must remember that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$. So when we divide the numerator by x , for $x < 0$ we get

$$\frac{1}{x} \sqrt{2x^2 + 1} = -\frac{1}{\sqrt{x^2}} \sqrt{2x^2 + 1} = -\sqrt{2 + \frac{1}{x^2}}$$

Therefore

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \rightarrow -\infty} \frac{-\sqrt{2 + \frac{1}{x^2}}}{3 - \frac{5}{x}} = \frac{-\sqrt{2 + \lim_{x \rightarrow -\infty} \frac{1}{x^2}}}{3 - 5 \lim_{x \rightarrow -\infty} \frac{1}{x}} = -\frac{\sqrt{2}}{3}$$

Thus the line $y = -\sqrt{2}/3$ is also a horizontal asymptote.

A vertical asymptote is likely to occur when the denominator, $3x - 5$, is 0, that is, when $x = \frac{5}{3}$. If x is close to $\frac{5}{3}$ and $x > \frac{5}{3}$, then the denominator is close to 0 and $3x - 5$ is positive. The numerator $\sqrt{2x^2 + 1}$ is always positive, so $f(x)$ is positive. Therefore

$$\lim_{x \rightarrow (5/3)^+} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \infty$$

If x is close to $\frac{5}{3}$ but $x < \frac{5}{3}$, then $3x - 5 < 0$ and so $f(x)$ is large negative. Thus

$$\lim_{x \rightarrow (5/3)^-} \frac{\sqrt{2x^2 + 1}}{3x - 5} = -\infty$$

The vertical asymptote is $x = \frac{5}{3}$. All three asymptotes are shown in Figure 8.

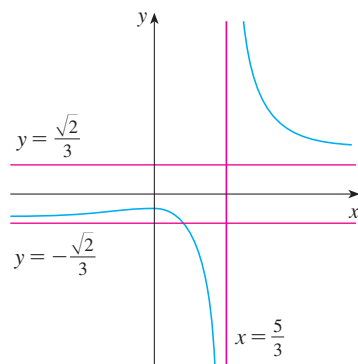


FIGURE 8

$$y = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

EXAMPLE 5 Compute $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x)$.

SOLUTION Because both $\sqrt{x^2 + 1}$ and x are large when x is large, it's difficult to see what happens to their difference, so we use algebra to rewrite the function. We first multiply numerator and denominator by the conjugate radical:

$$\begin{aligned} \lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x) &= \lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x) \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1} + x} \\ &= \lim_{x \rightarrow \infty} \frac{(x^2 + 1) - x^2}{\sqrt{x^2 + 1} + x} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x^2 + 1} + x} \end{aligned}$$

Notice that the denominator of this last expression ($\sqrt{x^2 + 1} + x$) becomes large as $x \rightarrow \infty$ (it's bigger than x). So

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x) = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x^2 + 1} + x} = 0$$

Figure 9 illustrates this result.

EXAMPLE 6 Evaluate $\lim_{x \rightarrow 2^+} \arctan\left(\frac{1}{x - 2}\right)$.

SOLUTION If we let $t = 1/(x - 2)$, we know that $t \rightarrow \infty$ as $x \rightarrow 2^+$. Therefore, by the second equation in [4], we have

$$\lim_{x \rightarrow 2^+} \arctan\left(\frac{1}{x - 2}\right) = \lim_{t \rightarrow \infty} \arctan t = \frac{\pi}{2}$$

The graph of the natural exponential function $y = e^x$ has the line $y = 0$ (the x -axis) as a horizontal asymptote. (The same is true of any exponential function with base $a > 1$.) In fact, from the graph in Figure 10 and the corresponding table of values, we see that

6

$$\lim_{x \rightarrow -\infty} e^x = 0$$

Notice that the values of e^x approach 0 very rapidly.

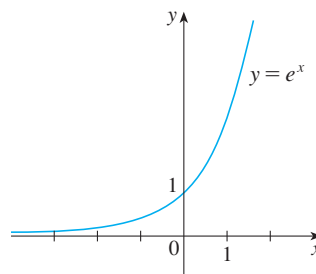


FIGURE 10

x	e^x
0	1.00000
-1	0.36788
-2	0.13534
-3	0.04979
-5	0.00674
-8	0.00034
-10	0.00005

We can think of the given function as having a denominator of 1.

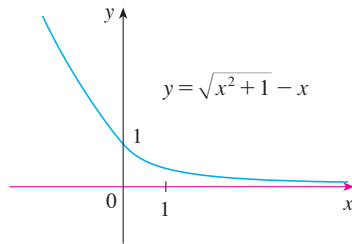


FIGURE 9

PS The problem-solving strategy for Examples 6 and 7 is *introducing something extra* (see page 75). Here, the something extra, the auxiliary aid, is the new variable t .

V EXAMPLE 7 Evaluate $\lim_{x \rightarrow 0^-} e^{1/x}$.

SOLUTION If we let $t = 1/x$, we know that $t \rightarrow -\infty$ as $x \rightarrow 0^-$. Therefore, by [6],

$$\lim_{x \rightarrow 0^-} e^{1/x} = \lim_{t \rightarrow -\infty} e^t = 0$$

(See Exercise 75.)

EXAMPLE 8 Evaluate $\lim_{x \rightarrow \infty} \sin x$.

SOLUTION As x increases, the values of $\sin x$ oscillate between 1 and -1 infinitely often and so they don't approach any definite number. Thus $\lim_{x \rightarrow \infty} \sin x$ does not exist.

Infinite Limits at Infinity

The notation

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x \rightarrow -\infty} f(x) = \infty \quad \lim_{x \rightarrow \infty} f(x) = -\infty \quad \lim_{x \rightarrow -\infty} f(x) = -\infty$$

EXAMPLE 9 Find $\lim_{x \rightarrow \infty} x^3$ and $\lim_{x \rightarrow -\infty} x^3$.

SOLUTION When x becomes large, x^3 also becomes large. For instance,

$$10^3 = 1000 \quad 100^3 = 1,000,000 \quad 1000^3 = 1,000,000,000$$

In fact, we can make x^3 as big as we like by taking x large enough. Therefore we can write

$$\lim_{x \rightarrow \infty} x^3 = \infty$$

Similarly, when x is large negative, so is x^3 . Thus

$$\lim_{x \rightarrow -\infty} x^3 = -\infty$$

These limit statements can also be seen from the graph of $y = x^3$ in Figure 11.

Looking at Figure 10 we see that

$$\lim_{x \rightarrow \infty} e^x = \infty$$

but, as Figure 12 demonstrates, $y = e^x$ becomes large as $x \rightarrow \infty$ at a much faster rate than $y = x^3$.

EXAMPLE 10 Find $\lim_{x \rightarrow \infty} (x^2 - x)$.

SOLUTION It would be **wrong** to write

$$\lim_{x \rightarrow \infty} (x^2 - x) = \lim_{x \rightarrow \infty} x^2 - \lim_{x \rightarrow \infty} x = \infty - \infty$$

The Limit Laws can't be applied to infinite limits because ∞ is not a number ($\infty - \infty$ can't be defined). However, we *can* write

$$\lim_{x \rightarrow \infty} (x^2 - x) = \lim_{x \rightarrow \infty} x(x - 1) = \infty$$

because both x and $x - 1$ become arbitrarily large and so their product does too.

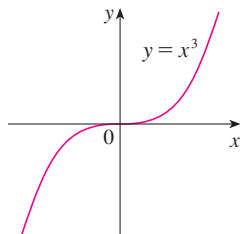


FIGURE 11

$$\lim_{x \rightarrow \infty} x^3 = \infty, \quad \lim_{x \rightarrow -\infty} x^3 = -\infty$$

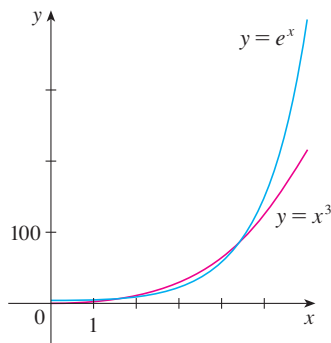


FIGURE 12

e^x is much larger than x^3 when x is large.

EXAMPLE 11 Find $\lim_{x \rightarrow \infty} \frac{x^2 + x}{3 - x}$.

SOLUTION As in Example 3, we divide the numerator and denominator by the highest power of x in the denominator, which is just x :

$$\lim_{x \rightarrow \infty} \frac{x^2 + x}{3 - x} = \lim_{x \rightarrow \infty} \frac{x + 1}{\frac{3}{x} - 1} = -\infty$$

because $x + 1 \rightarrow \infty$ and $3/x - 1 \rightarrow -1$ as $x \rightarrow \infty$.

The next example shows that by using infinite limits at infinity, together with intercepts, we can get a rough idea of the graph of a polynomial without having to plot a large number of points.

V EXAMPLE 12 Sketch the graph of $y = (x - 2)^4(x + 1)^3(x - 1)$ by finding its intercepts and its limits as $x \rightarrow \infty$ and as $x \rightarrow -\infty$.

SOLUTION The y -intercept is $f(0) = (-2)^4(1)^3(-1) = -16$ and the x -intercepts are found by setting $y = 0$: $x = 2, -1, 1$. Notice that since $(x - 2)^4$ is positive, the function doesn't change sign at 2; thus the graph doesn't cross the x -axis at 2. The graph crosses the axis at -1 and 1 .

When x is large positive, all three factors are large, so

$$\lim_{x \rightarrow \infty} (x - 2)^4(x + 1)^3(x - 1) = \infty$$

When x is large negative, the first factor is large positive and the second and third factors are both large negative, so

$$\lim_{x \rightarrow -\infty} (x - 2)^4(x + 1)^3(x - 1) = \infty$$

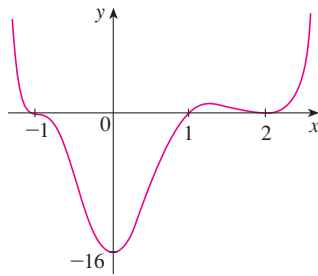


FIGURE 13
 $y = (x - 2)^4(x + 1)^3(x - 1)$

Combining this information, we give a rough sketch of the graph in Figure 13.

Precise Definitions

Definition 1 can be stated precisely as follows.

7 Definition Let f be a function defined on some interval (a, ∞) . Then

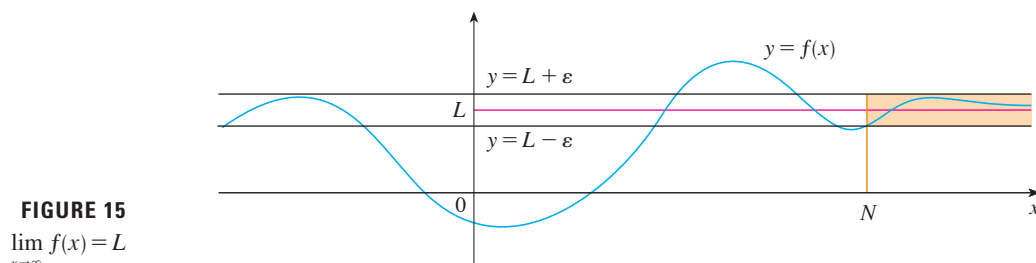
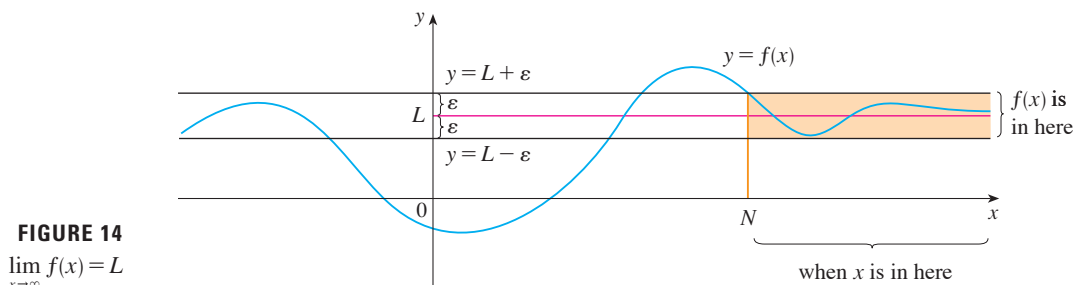
$$\lim_{x \rightarrow \infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if } x > N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

In words, this says that the values of $f(x)$ can be made arbitrarily close to L (within a distance ε , where ε is any positive number) by taking x sufficiently large (larger than N , where N depends on ε). Graphically it says that by choosing x large enough (larger than some number N) we can make the graph of f lie between the given horizontal lines

$y = L - \varepsilon$ and $y = L + \varepsilon$ as in Figure 14. This must be true no matter how small we choose ε . Figure 15 shows that if a smaller value of ε is chosen, then a larger value of N may be required.



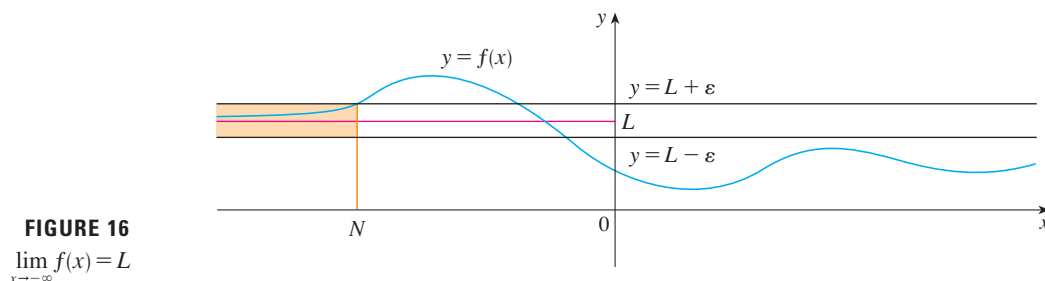
Similarly, a precise version of Definition 2 is given by Definition 8, which is illustrated in Figure 16.

8 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \rightarrow -\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if } x < N \quad \text{then} \quad |f(x) - L| < \varepsilon$$



In Example 3 we calculated that

$$\lim_{x \rightarrow \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{3}{5}$$

In the next example we use a graphing device to relate this statement to Definition 7 with $L = \frac{3}{5}$ and $\varepsilon = 0.1$.

TEC In Module 2.4/2.6 you can explore the precise definition of a limit both graphically and numerically.

EXAMPLE 13 Use a graph to find a number N such that

$$\text{if } x > N \quad \text{then} \quad \left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1$$

SOLUTION We rewrite the given inequality as

$$0.5 < \frac{3x^2 - x - 2}{5x^2 + 4x + 1} < 0.7$$

We need to determine the values of x for which the given curve lies between the horizontal lines $y = 0.5$ and $y = 0.7$. So we graph the curve and these lines in Figure 17. Then we use the cursor to estimate that the curve crosses the line $y = 0.5$ when $x \approx 6.7$. To the right of this number it seems that the curve stays between the lines $y = 0.5$ and $y = 0.7$. Rounding to be safe, we can say that

$$\text{if } x > 7 \quad \text{then} \quad \left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1$$

In other words, for $\varepsilon = 0.1$ we can choose $N = 7$ (or any larger number) in Definition 7.

EXAMPLE 14 Use Definition 7 to prove that $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$.

SOLUTION Given $\varepsilon > 0$, we want to find N such that

$$\text{if } x > N \quad \text{then} \quad \left| \frac{1}{x} - 0 \right| < \varepsilon$$

In computing the limit we may assume that $x > 0$. Then $1/x < \varepsilon \iff x > 1/\varepsilon$. Let's choose $N = 1/\varepsilon$. So

$$\text{if } x > N = \frac{1}{\varepsilon} \quad \text{then} \quad \left| \frac{1}{x} - 0 \right| = \frac{1}{x} < \varepsilon$$

Therefore, by Definition 7,

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

Figure 18 illustrates the proof by showing some values of ε and the corresponding values of N .

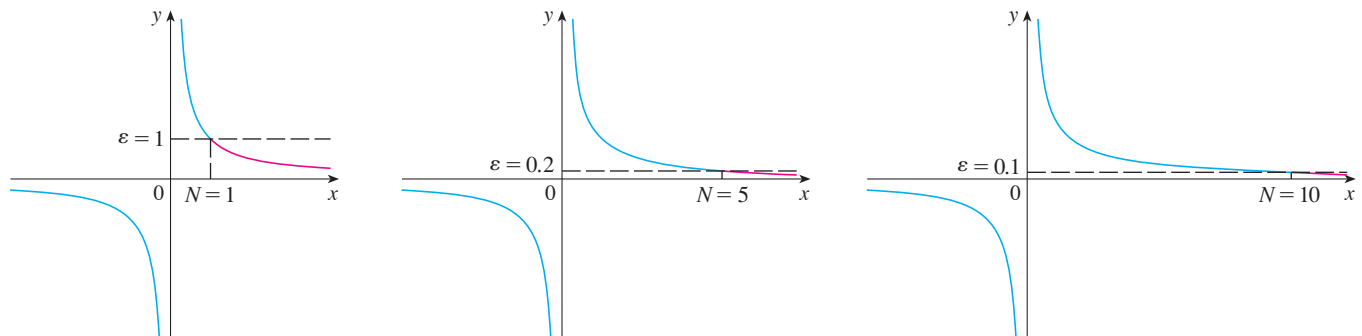


FIGURE 18

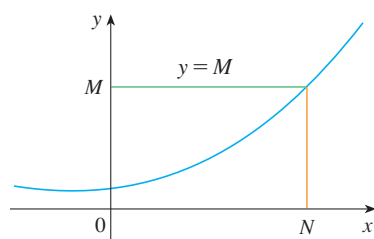


FIGURE 19
 $\lim_{x \rightarrow \infty} f(x) = \infty$

Finally we note that an infinite limit at infinity can be defined as follows. The geometric illustration is given in Figure 19.

9 Definition Let f be a function defined on some interval (a, ∞) . Then

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$\text{if } x > N \quad \text{then} \quad f(x) > M$$

Similar definitions apply when the symbol ∞ is replaced by $-\infty$. (See Exercise 74.)

2.6 Exercises

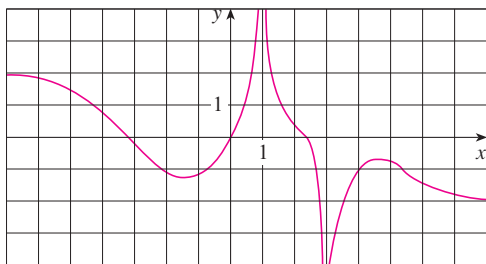
1. Explain in your own words the meaning of each of the following.

(a) $\lim_{x \rightarrow \infty} f(x) = 5$ (b) $\lim_{x \rightarrow -\infty} f(x) = 3$

2. (a) Can the graph of $y = f(x)$ intersect a vertical asymptote? Can it intersect a horizontal asymptote? Illustrate by sketching graphs.
 (b) How many horizontal asymptotes can the graph of $y = f(x)$ have? Sketch graphs to illustrate the possibilities.

3. For the function f whose graph is given, state the following.

(a) $\lim_{x \rightarrow \infty} f(x)$ (b) $\lim_{x \rightarrow -\infty} f(x)$
 (c) $\lim_{x \rightarrow 1} f(x)$ (d) $\lim_{x \rightarrow 3} f(x)$
 (e) The equations of the asymptotes

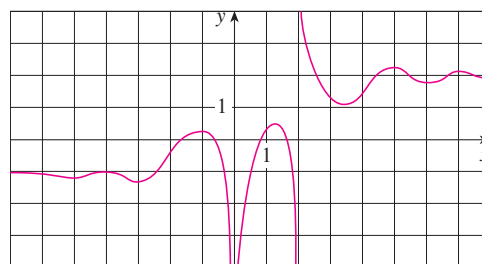


4. For the function g whose graph is given, state the following.

(a) $\lim_{x \rightarrow \infty} g(x)$ (b) $\lim_{x \rightarrow -\infty} g(x)$
 (c) $\lim_{x \rightarrow 0} g(x)$ (d) $\lim_{x \rightarrow 2^-} g(x)$


(e) $\lim_{x \rightarrow 2^+} g(x)$

(f) The equations of the asymptotes



- 5–10 Sketch the graph of an example of a function f that satisfies all of the given conditions.

5. $\lim_{x \rightarrow 0} f(x) = -\infty$, $\lim_{x \rightarrow -\infty} f(x) = 5$, $\lim_{x \rightarrow \infty} f(x) = -5$
 6. $\lim_{x \rightarrow 2} f(x) = \infty$, $\lim_{x \rightarrow -2^+} f(x) = \infty$, $\lim_{x \rightarrow -2^-} f(x) = -\infty$,
 $\lim_{x \rightarrow -\infty} f(x) = 0$, $\lim_{x \rightarrow \infty} f(x) = 0$, $f(0) = 0$
 7. $\lim_{x \rightarrow 2} f(x) = -\infty$, $\lim_{x \rightarrow \infty} f(x) = \infty$, $\lim_{x \rightarrow -\infty} f(x) = 0$,
 $\lim_{x \rightarrow 0^+} f(x) = \infty$, $\lim_{x \rightarrow 0^-} f(x) = -\infty$
 8. $\lim_{x \rightarrow \infty} f(x) = 3$, $\lim_{x \rightarrow 2^-} f(x) = \infty$, $\lim_{x \rightarrow 2^+} f(x) = -\infty$, f is odd
 9. $f(0) = 3$, $\lim_{x \rightarrow 0^-} f(x) = 4$, $\lim_{x \rightarrow 0^+} f(x) = 2$,
 $\lim_{x \rightarrow -\infty} f(x) = -\infty$, $\lim_{x \rightarrow 4^-} f(x) = -\infty$, $\lim_{x \rightarrow 4^+} f(x) = \infty$,
 $\lim_{x \rightarrow \infty} f(x) = 3$
 10. $\lim_{x \rightarrow 3} f(x) = -\infty$, $\lim_{x \rightarrow \infty} f(x) = 2$, $f(0) = 0$, f is even

-  11. Guess the value of the limit

$$\lim_{x \rightarrow \infty} \frac{x^2}{2^x}$$

by evaluating the function $f(x) = x^2/2^x$ for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50$, and 100. Then use a graph of f to support your guess.

-  12. (a) Use a graph of

$$f(x) = \left(1 - \frac{2}{x}\right)^x$$

to estimate the value of $\lim_{x \rightarrow \infty} f(x)$ correct to two decimal places.

- (b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

13–14 Evaluate the limit and justify each step by indicating the appropriate properties of limits.

13. $\lim_{x \rightarrow \infty} \frac{3x^2 - x + 4}{2x^2 + 5x - 8}$

14. $\lim_{x \rightarrow \infty} \sqrt{\frac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}}$

15–38 Find the limit or show that it does not exist.

15. $\lim_{x \rightarrow \infty} \frac{3x - 2}{2x + 1}$

16. $\lim_{x \rightarrow \infty} \frac{1 - x^2}{x^3 - x + 1}$

17. $\lim_{x \rightarrow -\infty} \frac{x - 2}{x^2 + 1}$

18. $\lim_{x \rightarrow -\infty} \frac{4x^3 + 6x^2 - 2}{2x^3 - 4x + 5}$

19. $\lim_{t \rightarrow \infty} \frac{\sqrt{t} + t^2}{2t - t^2}$

20. $\lim_{t \rightarrow \infty} \frac{t - t\sqrt{t}}{2t^{3/2} + 3t - 5}$

21. $\lim_{x \rightarrow \infty} \frac{(2x^2 + 1)^2}{(x - 1)^2(x^2 + x)}$

22. $\lim_{x \rightarrow \infty} \frac{x^2}{\sqrt{x^4 + 1}}$

23. $\lim_{x \rightarrow \infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1}$

24. $\lim_{x \rightarrow -\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1}$

25. $\lim_{x \rightarrow \infty} (\sqrt{9x^2 + x} - 3x)$

26. $\lim_{x \rightarrow -\infty} (x + \sqrt{x^2 + 2x})$

27. $\lim_{x \rightarrow \infty} (\sqrt{x^2 + ax} - \sqrt{x^2 + bx})$

28. $\lim_{x \rightarrow \infty} \sqrt{x^2 + 1}$

29. $\lim_{x \rightarrow \infty} \frac{x^4 - 3x^2 + x}{x^3 - x + 2}$

30. $\lim_{x \rightarrow \infty} (e^{-x} + 2 \cos 3x)$

31. $\lim_{x \rightarrow -\infty} (x^4 + x^5)$

32. $\lim_{x \rightarrow -\infty} \frac{1 + x^6}{x^4 + 1}$

33. $\lim_{x \rightarrow \infty} \arctan(e^x)$

34. $\lim_{x \rightarrow \infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

35. $\lim_{x \rightarrow \infty} \frac{1 - e^x}{1 + 2e^x}$

36. $\lim_{x \rightarrow \infty} \frac{\sin^2 x}{x^2 + 1}$

37. $\lim_{x \rightarrow \infty} (e^{-2x} \cos x)$

38. $\lim_{x \rightarrow 0^+} \tan^{-1}(\ln x)$

-  39. (a) Estimate the value of

$$\lim_{x \rightarrow -\infty} (\sqrt{x^2 + x + 1} + x)$$

by graphing the function $f(x) = \sqrt{x^2 + x + 1} + x$.

- (b) Use a table of values of $f(x)$ to guess the value of the limit.
(c) Prove that your guess is correct.

-  40. (a) Use a graph of

$$f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}$$

to estimate the value of $\lim_{x \rightarrow \infty} f(x)$ to one decimal place.

- (b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.
(c) Find the exact value of the limit.

41–46 Find the horizontal and vertical asymptotes of each curve. If you have a graphing device, check your work by graphing the curve and estimating the asymptotes.

41. $y = \frac{2x + 1}{x - 2}$

42. $y = \frac{x^2 + 1}{2x^2 - 3x - 2}$

43. $y = \frac{2x^2 + x - 1}{x^2 + x - 2}$

44. $y = \frac{1 + x^4}{x^2 - x^4}$

45. $y = \frac{x^3 - x}{x^2 - 6x + 5}$

46. $y = \frac{2e^x}{e^x - 5}$

-  47. Estimate the horizontal asymptote of the function

$$f(x) = \frac{3x^3 + 500x^2}{x^3 + 500x^2 + 100x + 2000}$$

by graphing f for $-10 \leq x \leq 10$. Then calculate the equation of the asymptote by evaluating the limit. How do you explain the discrepancy?

-  48. (a) Graph the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

How many horizontal and vertical asymptotes do you observe? Use the graph to estimate the values of the limits

$$\lim_{x \rightarrow \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

- (b) By calculating values of $f(x)$, give numerical estimates of the limits in part (a).
(c) Calculate the exact values of the limits in part (a). Did you get the same value or different values for these two limits? [In view of your answer to part (a), you might have to check your calculation for the second limit.]

49. Find a formula for a function f that satisfies the following conditions:

$$\lim_{x \rightarrow \pm\infty} f(x) = 0, \quad \lim_{x \rightarrow 0} f(x) = -\infty, \quad f(2) = 0,$$

$$\lim_{x \rightarrow 3^-} f(x) = \infty, \quad \lim_{x \rightarrow 3^+} f(x) = -\infty$$

50. Find a formula for a function that has vertical asymptotes $x = 1$ and $x = 3$ and horizontal asymptote $y = 1$.
51. A function f is a ratio of quadratic functions and has a vertical asymptote $x = 4$ and just one x -intercept, $x = 1$. It is known that f has a removable discontinuity at $x = -1$ and $\lim_{x \rightarrow -1} f(x) = 2$. Evaluate
- (a) $f(0)$ (b) $\lim_{x \rightarrow \infty} f(x)$

52–56 Find the limits as $x \rightarrow \infty$ and as $x \rightarrow -\infty$. Use this information, together with intercepts, to give a rough sketch of the graph as in Example 12.

52. $y = 2x^3 - x^4$


53. $y = x^4 - x^6$


54. $y = x^3(x + 2)^2(x - 1)$

55. $y = (3 - x)(1 + x)^2(1 - x)^4$

56. $y = x^2(x^2 - 1)^2(x + 2)$

57. (a) Use the Squeeze Theorem to evaluate $\lim_{x \rightarrow \infty} \frac{\sin x}{x}$.

 (b) Graph $f(x) = (\sin x)/x$. How many times does the graph cross the asymptote?

 58. By the *end behavior* of a function we mean the behavior of its values as $x \rightarrow \infty$ and as $x \rightarrow -\infty$.

- (a) Describe and compare the end behavior of the functions

$$P(x) = 3x^5 - 5x^3 + 2x \quad Q(x) = 3x^5$$

by graphing both functions in the viewing rectangles $[-2, 2]$ by $[-2, 2]$ and $[-10, 10]$ by $[-10,000, 10,000]$.

- (b) Two functions are said to have the *same end behavior* if their ratio approaches 1 as $x \rightarrow \infty$. Show that P and Q have the same end behavior.

59. Let P and Q be polynomials. Find

$$\lim_{x \rightarrow \infty} \frac{P(x)}{Q(x)}$$

- if the degree of P is (a) less than the degree of Q and
(b) greater than the degree of Q .

60. Make a rough sketch of the curve $y = x^n$ (n an integer) for the following five cases:

- (i) $n = 0$ (ii) $n > 0, n$ odd
(iii) $n > 0, n$ even (iv) $n < 0, n$ odd
(v) $n < 0, n$ even

Then use these sketches to find the following limits.

- (a) $\lim_{x \rightarrow 0^+} x^n$ (b) $\lim_{x \rightarrow 0^-} x^n$
(c) $\lim_{x \rightarrow \infty} x^n$ (d) $\lim_{x \rightarrow -\infty} x^n$

61. Find $\lim_{x \rightarrow \infty} f(x)$ if, for all $x > 1$,

$$\frac{10e^x - 21}{2e^x} < f(x) < \frac{5\sqrt{x}}{\sqrt{x-1}}$$

62. (a) A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that the concentration of salt after t minutes (in grams per liter) is

$$C(t) = \frac{30t}{200 + t}$$

- (b) What happens to the concentration as $t \rightarrow \infty$?

63. In Chapter 9 we will be able to show, under certain assumptions, that the velocity $v(t)$ of a falling raindrop at time t is

$$v(t) = v^*(1 - e^{-gt/v^*})$$

where g is the acceleration due to gravity and v^* is the *terminal velocity* of the raindrop.

- (a) Find $\lim_{t \rightarrow \infty} v(t)$.



- (b) Graph $v(t)$ if $v^* = 1$ m/s and $g = 9.8$ m/s². How long does it take for the velocity of the raindrop to reach 99% of its terminal velocity?



64. (a) By graphing $y = e^{-x/10}$ and $y = 0.1$ on a common screen, discover how large you need to make x so that $e^{-x/10} < 0.1$.
(b) Can you solve part (a) without using a graphing device?



65. Use a graph to find a number N such that

$$\text{if } x > N \quad \text{then} \quad \left| \frac{3x^2 + 1}{2x^2 + x + 1} - 1.5 \right| < 0.05$$



66. For the limit

$$\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 1}}{x + 1} = 2$$

illustrate Definition 7 by finding values of N that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.



67. For the limit

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{x + 1} = -2$$

illustrate Definition 8 by finding values of N that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.



68. For the limit

$$\lim_{x \rightarrow \infty} \frac{2x + 1}{\sqrt{x + 1}} = \infty$$

illustrate Definition 9 by finding a value of N that corresponds to $M = 100$.

69. (a) How large do we have to take x so that $1/x^2 < 0.0001$?
 (b) Taking $r = 2$ in Theorem 5, we have the statement

$$\lim_{x \rightarrow \infty} \frac{1}{x^2} = 0$$

Prove this directly using Definition 7.

70. (a) How large do we have to take x so that $1/\sqrt{x} < 0.0001$?
 (b) Taking $r = \frac{1}{2}$ in Theorem 5, we have the statement

$$\lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} = 0$$

Prove this directly using Definition 7.

71. Use Definition 8 to prove that $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$.

72. Prove, using Definition 9, that $\lim_{x \rightarrow \infty} x^3 = \infty$.

73. Use Definition 9 to prove that $\lim_{x \rightarrow \infty} e^x = \infty$.

74. Formulate a precise definition of

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

Then use your definition to prove that

$$\lim_{x \rightarrow -\infty} (1 + x^3) = -\infty$$

75. Prove that

$$\lim_{x \rightarrow \infty} f(x) = \lim_{t \rightarrow 0^+} f(1/t)$$

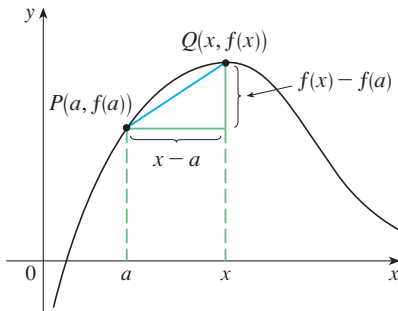
and

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{t \rightarrow 0^-} f(1/t)$$

if these limits exist.

2.7 Derivatives and Rates of Change

The problem of finding the tangent line to a curve and the problem of finding the velocity of an object both involve finding the same type of limit, as we saw in Section 2.1. This special type of limit is called a *derivative* and we will see that it can be interpreted as a rate of change in any of the sciences or engineering.



Tangents

If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ :

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let Q approach P along the curve C by letting x approach a . If m_{PQ} approaches a number m , then we define the *tangent* t to be the line through P with slope m . (This amounts to saying that the tangent line is the limiting position of the secant line PQ as Q approaches P . See Figure 1.)

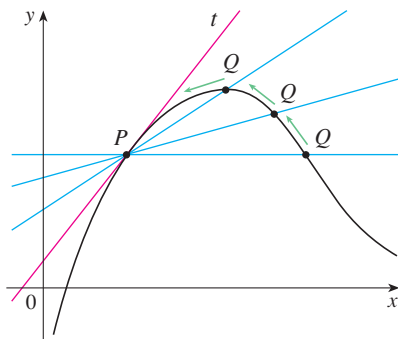


FIGURE 1

1 Definition The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

In our first example we confirm the guess we made in Example 1 in Section 2.1.

V EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

SOLUTION Here we have $a = 1$ and $f(x) = x^2$, so the slope is

$$\begin{aligned} m &= \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{x - 1} \\ &= \lim_{x \rightarrow 1} (x + 1) = 1 + 1 = 2 \end{aligned}$$

Point-slope form for a line through the point (x_1, y_1) with slope m :

$$y - y_1 = m(x - x_1)$$

Using the point-slope form of the equation of a line, we find that an equation of the tangent line at $(1, 1)$ is

$$y - 1 = 2(x - 1) \quad \text{or} \quad y = 2x - 1$$

We sometimes refer to the slope of the tangent line to a curve at a point as the **slope of the curve** at the point. The idea is that if we zoom in far enough toward the point, the curve looks almost like a straight line. Figure 2 illustrates this procedure for the curve $y = x^2$ in Example 1. The more we zoom in, the more the parabola looks like a line. In other words, the curve becomes almost indistinguishable from its tangent line.

TEC Visual 2.7 shows an animation of Figure 2.

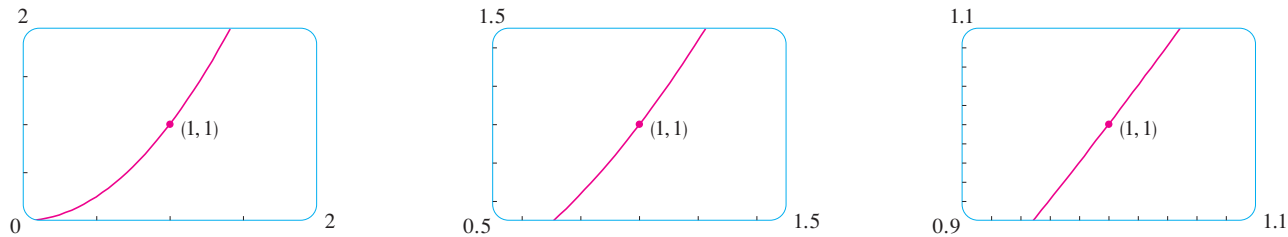


FIGURE 2 Zooming in toward the point $(1, 1)$ on the parabola $y = x^2$

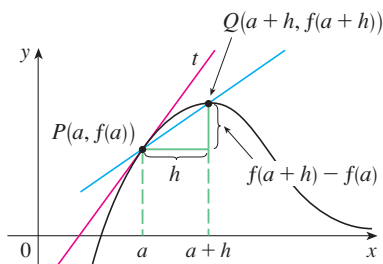


FIGURE 3

There is another expression for the slope of a tangent line that is sometimes easier to use. If $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a + h) - f(a)}{h}$$

(See Figure 3 where the case $h > 0$ is illustrated and Q is to the right of P . If it happened that $h < 0$, however, Q would be to the left of P .)

Notice that as x approaches a , h approaches 0 (because $h = x - a$) and so the expression for the slope of the tangent line in Definition 1 becomes

2

$$m = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

EXAMPLE 2 Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point $(3, 1)$.

SOLUTION Let $f(x) = 3/x$. Then the slope of the tangent at $(3, 1)$ is

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \rightarrow 0} \frac{\frac{3}{3+h} - 1}{h} = \lim_{h \rightarrow 0} \frac{\frac{3 - (3+h)}{3+h}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{h(3+h)} = \lim_{h \rightarrow 0} -\frac{1}{3+h} = -\frac{1}{3} \end{aligned}$$

Therefore an equation of the tangent at the point $(3, 1)$ is

$$y - 1 = -\frac{1}{3}(x - 3)$$

which simplifies to

$$x + 3y - 6 = 0$$

The hyperbola and its tangent are shown in Figure 4.

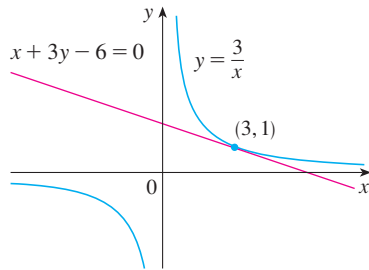


FIGURE 4

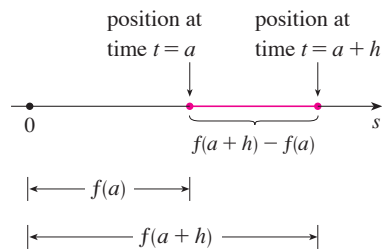


FIGURE 5

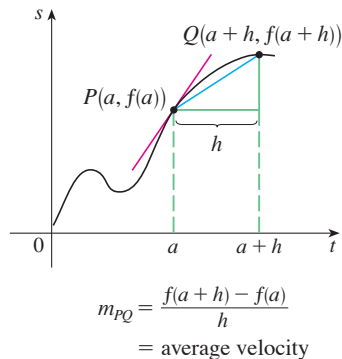


FIGURE 6

Velocities

In Section 2.1 we investigated the motion of a ball dropped from the CN Tower and defined its velocity to be the limiting value of average velocities over shorter and shorter time periods.

In general, suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t . The function f that describes the motion is called the **position function** of the object. In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) - f(a)$. (See Figure 5.) The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a + h) - f(a)}{h}$$

which is the same as the slope of the secant line PQ in Figure 6.

Now suppose we compute the average velocities over shorter and shorter time intervals $[a, a + h]$. In other words, we let h approach 0. As in the example of the falling ball, we define the **velocity** (or **instantaneous velocity**) $v(a)$ at time $t = a$ to be the limit of these average velocities:

3

$$v(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

This means that the velocity at time $t = a$ is equal to the slope of the tangent line at P (compare Equations 2 and 3).

Now that we know how to compute limits, let's reconsider the problem of the falling ball.

V EXAMPLE 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.

- What is the velocity of the ball after 5 seconds?
- How fast is the ball traveling when it hits the ground?

SOLUTION We will need to find the velocity both when $t = 5$ and when the ball hits the ground, so it's efficient to start by finding the velocity at a general time . Using the

Recall from Section 2.1: The distance (in meters) fallen after t seconds is $4.9t^2$.

equation of motion $s = f(t) = 4.9t^2$, we have

$$\begin{aligned} v(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{4.9(a+h)^2 - 4.9a^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{4.9(a^2 + 2ah + h^2 - a^2)}{h} = \lim_{h \rightarrow 0} \frac{4.9(2ah + h^2)}{h} \\ &= \lim_{h \rightarrow 0} 4.9(2a + h) = 9.8a \end{aligned}$$

(a) The velocity after 5 s is $v(5) = (9.8)(5) = 49$ m/s.

(b) Since the observation deck is 450 m above the ground, the ball will hit the ground at the time t_1 when $s(t_1) = 450$, that is,

$$4.9t_1^2 = 450$$

This gives

$$t_1^2 = \frac{450}{4.9} \quad \text{and} \quad t_1 = \sqrt{\frac{450}{4.9}} \approx 9.6 \text{ s}$$

The velocity of the ball as it hits the ground is therefore

$$v(t_1) = 9.8t_1 = 9.8\sqrt{\frac{450}{4.9}} \approx 94 \text{ m/s}$$

Derivatives

We have seen that the same type of limit arises in finding the slope of a tangent line (Equation 2) or the velocity of an object (Equation 3). In fact, limits of the form

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

arise whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics. Since this type of limit occurs so widely, it is given a special name and notation.

4 Definition The **derivative of a function f at a number a** , denoted by $f'(a)$, is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

$f'(a)$ is read “ f prime of a .”

If we write $x = a + h$, then we have $h = x - a$ and h approaches 0 if and only if x approaches a . Therefore an equivalent way of stating the definition of the derivative, as we saw in finding tangent lines, is

5

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

V EXAMPLE 4 Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a .

SOLUTION From Definition 4 we have

$$\begin{aligned}
 f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h \rightarrow 0} (2a + h - 8) \\
 &= 2a - 8
 \end{aligned}$$

We defined the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ to be the line that passes through P and has slope m given by Equation 1 or 2. Since, by Definition 4, this is the same as the derivative $f'(a)$, we can now say the following.

The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a .

If we use the point-slope form of the equation of a line, we can write an equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$

V EXAMPLE 5 Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

SOLUTION From Example 4 we know that the derivative of $f(x) = x^2 - 8x + 9$ at the number a is $f'(a) = 2a - 8$. Therefore the slope of the tangent line at $(3, -6)$ is $f'(3) = 2(3) - 8 = -2$. Thus an equation of the tangent line, shown in Figure 7, is

$$y - (-6) = (-2)(x - 3) \quad \text{or} \quad y = -2x$$

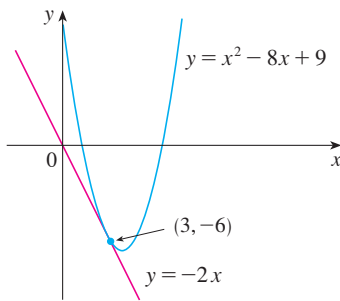


FIGURE 7

Rates of Change

Suppose y is a quantity that depends on another quantity x . Thus y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2 , then the change in x (also called the **increment** of x) is

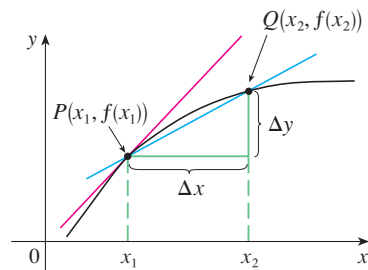
$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



average rate of change = m_{PQ}
 instantaneous rate of change =
 slope of tangent at P

FIGURE 8

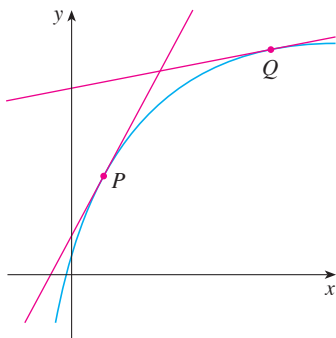


FIGURE 9
 The y -values are changing rapidly at P and slowly at Q .

is called the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in Figure 8.

By analogy with velocity, we consider the average rate of change over smaller and smaller intervals by letting x_2 approach x_1 and therefore letting Δx approach 0. The limit of these average rates of change is called the **(instantaneous) rate of change of y with respect to x** at $x = x_1$, which is interpreted as the slope of the tangent to the curve $y = f(x)$ at $P(x_1, f(x_1))$:

$$\boxed{6} \quad \text{instantaneous rate of change} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \rightarrow x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

We recognize this limit as being the derivative $f'(x_1)$.

We know that one interpretation of the derivative $f'(a)$ is as the slope of the tangent line to the curve $y = f(x)$ when $x = a$. We now have a second interpretation:

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

The connection with the first interpretation is that if we sketch the curve $y = f(x)$, then the instantaneous rate of change is the slope of the tangent to this curve at the point where $x = a$. This means that when the derivative is large (and therefore the curve is steep, as at the point P in Figure 9), the y -values change rapidly. When the derivative is small, the curve is relatively flat (as at point Q) and the y -values change slowly.

In particular, if $s = f(t)$ is the position function of a particle that moves along a straight line, then $f'(a)$ is the rate of change of the displacement s with respect to the time t . In other words, $f'(a)$ is the *velocity of the particle at time $t = a$* . The **speed** of the particle is the absolute value of the velocity, that is, $|f'(a)|$.

In the next example we discuss the meaning of the derivative of a function that is defined verbally.

V EXAMPLE 6 A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x yards of this fabric is $C = f(x)$ dollars.

- What is the meaning of the derivative $f'(x)$? What are its units?
- In practical terms, what does it mean to say that $f'(1000) = 9$?
- Which do you think is greater, $f'(50)$ or $f'(500)$? What about $f'(5000)$?

SOLUTION

(a) The derivative $f'(x)$ is the instantaneous rate of change of C with respect to x ; that is, $f'(x)$ means the rate of change of the production cost with respect to the number of yards produced. (Economists call this rate of change the *marginal cost*. This idea is discussed in more detail in Sections 3.7 and 4.7.)

Because

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta C}{\Delta x}$$

the units for $f'(x)$ are the same as the units for the difference quotient $\Delta C/\Delta x$. Since ΔC is measured in dollars and Δx in yards, it follows that the units for $f'(x)$ are dollars per yard.

(b) The statement that $f'(1000) = 9$ means that, after 1000 yards of fabric have been manufactured, the rate at which the production cost is increasing is \$9/yard. (When $x = 1000$, C is increasing 9 times as fast as x .)

Since $\Delta x = 1$ is small compared with $x = 1000$, we could use the approximation

$$f'(1000) \approx \frac{\Delta C}{\Delta x} = \frac{\Delta C}{1} = \Delta C$$

and say that the cost of manufacturing the 1000th yard (or the 1001st) is about \$9.

(c) The rate at which the production cost is increasing (per yard) is probably lower when $x = 500$ than when $x = 50$ (the cost of making the 500th yard is less than the cost of the 50th yard) because of economies of scale. (The manufacturer makes more efficient use of the fixed costs of production.) So

$$f'(50) > f'(500)$$

But, as production expands, the resulting large-scale operation might become inefficient and there might be overtime costs. Thus it is possible that the rate of increase of costs will eventually start to rise. So it may happen that

$$f'(5000) > f'(500)$$

In the following example we estimate the rate of change of the national debt with respect to time. Here the function is defined not by a formula but by a table of values.

t	$D(t)$
1980	930.2
1985	1945.9
1990	3233.3
1995	4974.0
2000	5674.2
2005	7932.7

V EXAMPLE 7 Let $D(t)$ be the US national debt at time t . The table in the margin gives approximate values of this function by providing end of year estimates, in billions of dollars, from 1980 to 2005. Interpret and estimate the value of $D'(1990)$.

SOLUTION The derivative $D'(1990)$ means the rate of change of D with respect to t when $t = 1990$, that is, the rate of increase of the national debt in 1990.

According to Equation 5,

$$D'(1990) = \lim_{t \rightarrow 1990} \frac{D(t) - D(1990)}{t - 1990}$$

So we compute and tabulate values of the difference quotient (the average rates of change) as follows.

t	$\frac{D(t) - D(1990)}{t - 1990}$
1980	230.31
1985	257.48
1995	348.14
2000	244.09
2005	313.29

From this table we see that $D'(1990)$ lies somewhere between 257.48 and 348.14 billion dollars per year. [Here we are making the reasonable assumption that the debt didn't fluctuate wildly between 1980 and 2000.] We estimate that the rate of increase of the national debt of the United States in 1990 was the average of these two numbers, namely

$$D'(1990) \approx 303 \text{ billion dollars per year}$$

Another method would be to plot the debt function and estimate the slope of the tangent line when $t = 1990$.

Here we are assuming that the cost function is well behaved; in other words, $C(x)$ doesn't oscillate rapidly near $x = 1000$.

A Note on Units

The units for the average rate of change $\Delta D / \Delta t$ are the units for ΔD divided by the units for Δt , namely, billions of dollars per year. The instantaneous rate of change is the limit of the average rates of change, so it is measured in the same units: billions of dollars per year.


In Examples 3, 6, and 7 we saw three specific examples of rates of change: the velocity of an object is the rate of change of displacement with respect to time; marginal cost is the rate of change of production cost with respect to the number of items produced; the rate of change of the debt with respect to time is of interest in economics. Here is a small sample of other rates of change: In physics, the rate of change of work with respect to time is called *power*. Chemists who study a chemical reaction are interested in the rate of change in the concentration of a reactant with respect to time (called the *rate of reaction*). A biologist is interested in the rate of change of the population of a colony of bacteria with respect to time. In fact, the computation of rates of change is important in all of the natural sciences, in engineering, and even in the social sciences. Further examples will be given in Section 3.7.

All these rates of change are derivatives and can therefore be interpreted as slopes of tangents. This gives added significance to the solution of the tangent problem. Whenever we solve a problem involving tangent lines, we are not just solving a problem in geometry. We are also implicitly solving a great variety of problems involving rates of change in science and engineering.

2.7 Exercises


1. A curve has equation $y = f(x)$.

- (a) Write an expression for the slope of the secant line through the points $P(3, f(3))$ and $Q(x, f(x))$.
(b) Write an expression for the slope of the tangent line at P .

-  2. Graph the curve $y = e^x$ in the viewing rectangles $[-1, 1]$ by $[0, 2]$, $[-0.5, 0.5]$ by $[0.5, 1.5]$, and $[-0.1, 0.1]$ by $[0.9, 1.1]$. What do you notice about the curve as you zoom in toward the point $(0, 1)$?


3. (a) Find the slope of the tangent line to the parabola $y = 4x - x^2$ at the point $(1, 3)$
(i) using Definition 1 (ii) using Equation 2

- (b) Find an equation of the tangent line in part (a).

-  (c) Graph the parabola and the tangent line. As a check on your work, zoom in toward the point $(1, 3)$ until the parabola and the tangent line are indistinguishable.

4. (a) Find the slope of the tangent line to the curve $y = x - x^3$ at the point $(1, 0)$
(i) using Definition 1 (ii) using Equation 2

- (b) Find an equation of the tangent line in part (a).

-  (c) Graph the curve and the tangent line in successively smaller viewing rectangles centered at $(1, 0)$ until the curve and the line appear to coincide.

5–8 Find an equation of the tangent line to the curve at the given point.

5. $y = 4x - 3x^2$, $(2, -4)$

6. $y = x^3 - 3x + 1$, $(2, 3)$

7. $y = \sqrt{x}$, $(1, 1)$

8. $y = \frac{2x + 1}{x + 2}$, $(1, 1)$

9. (a) Find the slope of the tangent to the curve $y = 3 + 4x^2 - 2x^3$ at the point where $x = a$.

- (b) Find equations of the tangent lines at the points $(1, 5)$ and $(2, 3)$.

-  (c) Graph the curve and both tangents on a common screen.

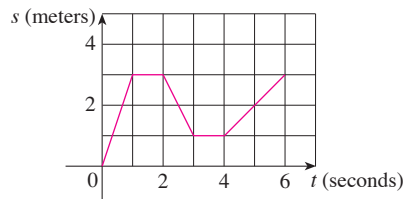
10. (a) Find the slope of the tangent to the curve $y = 1/\sqrt{x}$ at the point where $x = a$.

- (b) Find equations of the tangent lines at the points $(1, 1)$ and $(4, \frac{1}{2})$.

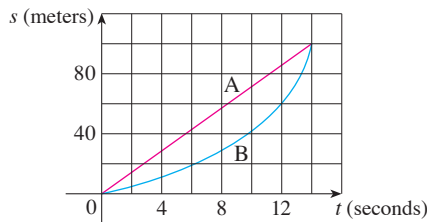
-  (c) Graph the curve and both tangents on a common screen.

11. (a) A particle starts by moving to the right along a horizontal line; the graph of its position function is shown. When is the particle moving to the right? Moving to the left? Standing still?

- (b) Draw a graph of the velocity function.

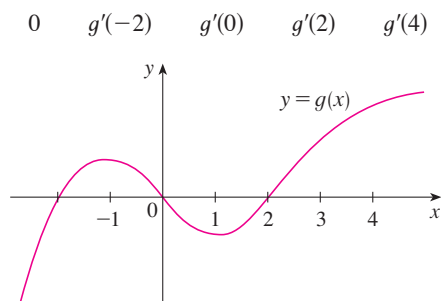


12. Shown are graphs of the position functions of two runners, A and B, who run a 100-m race and finish in a tie.



- (a) Describe and compare how the runners run the race.
 (b) At what time is the distance between the runners the greatest?
 (c) At what time do they have the same velocity?
13. If a ball is thrown into the air with a velocity of 40 ft/s, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. Find the velocity when $t = 2$.
14. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height (in meters) after t seconds is given by $H = 10t - 1.86t^2$.
 (a) Find the velocity of the rock after one second.
 (b) Find the velocity of the rock when $t = a$.
 (c) When will the rock hit the surface?
 (d) With what velocity will the rock hit the surface?
15. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $s = 1/t^2$, where t is measured in seconds. Find the velocity of the particle at times $t = a$, $t = 1$, $t = 2$, and $t = 3$.
16. The displacement (in meters) of a particle moving in a straight line is given by $s = t^2 - 8t + 18$, where t is measured in seconds.
 (a) Find the average velocity over each time interval:
 (i) $[3, 4]$ (ii) $[3.5, 4]$
 (iii) $[4, 5]$ (iv) $[4, 4.5]$
 (b) Find the instantaneous velocity when $t = 4$.
 (c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a) and the tangent line whose slope is the instantaneous velocity in part (b).

17. For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:



18. Find an equation of the tangent line to the graph of $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$.
19. If an equation of the tangent line to the curve $y = f(x)$ at the point where $a = 2$ is $y = 4x - 5$, find $f(2)$ and $f'(2)$.
20. If the tangent line to $y = f(x)$ at $(4, 3)$ passes through the point $(0, 2)$, find $f(4)$ and $f'(4)$.
21. Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = 3$, $f'(1) = 0$, and $f'(2) = -1$.

22. Sketch the graph of a function g for which $g(0) = g(2) = g(4) = 0$, $g'(1) = g'(3) = 0$, $g'(0) = g'(4) = 1$, $g'(2) = -1$, $\lim_{x \rightarrow \infty} g(x) = \infty$, and $\lim_{x \rightarrow -\infty} g(x) = -\infty$.
23. If $f(x) = 3x^2 - x^3$, find $f'(1)$ and use it to find an equation of the tangent line to the curve $y = 3x^2 - x^3$ at the point $(1, 2)$.
24. If $g(x) = x^4 - 2$, find $g'(1)$ and use it to find an equation of the tangent line to the curve $y = x^4 - 2$ at the point $(1, -1)$.
25. (a) If $F(x) = 5x/(1 + x^2)$, find $F'(2)$ and use it to find an equation of the tangent line to the curve $y = 5x/(1 + x^2)$ at the point $(2, 2)$.
 (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.
26. (a) If $G(x) = 4x^2 - x^3$, find $G'(a)$ and use it to find equations of the tangent lines to the curve $y = 4x^2 - x^3$ at the points $(2, 8)$ and $(3, 9)$.
 (b) Illustrate part (a) by graphing the curve and the tangent lines on the same screen.

27–32 Find $f'(a)$.

27. $f(x) = 3x^2 - 4x + 1$

28. $f(t) = 2t^3 + t$

29. $f(t) = \frac{2t + 1}{t + 3}$

30. $f(x) = x^{-2}$

31. $f(x) = \sqrt{1 - 2x}$

32. $f(x) = \frac{4}{\sqrt{1 - x}}$

33–38 Each limit represents the derivative of some function f at some number a . State such an f and a in each case.

33. $\lim_{h \rightarrow 0} \frac{(1 + h)^{10} - 1}{h}$

34. $\lim_{h \rightarrow 0} \frac{\sqrt[4]{16 + h} - 2}{h}$

35. $\lim_{x \rightarrow 5} \frac{2^x - 32}{x - 5}$

36. $\lim_{x \rightarrow \pi/4} \frac{\tan x - 1}{x - \pi/4}$

37. $\lim_{h \rightarrow 0} \frac{\cos(\pi + h) + 1}{h}$

38. $\lim_{t \rightarrow 1} \frac{t^4 + t - 2}{t - 1}$

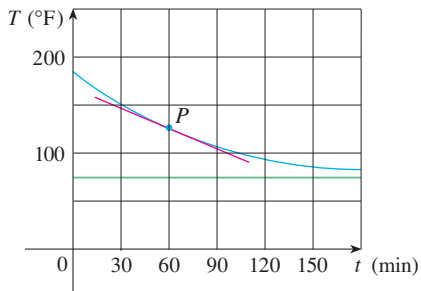
39–40 A particle moves along a straight line with equation of motion $s = f(t)$, where s is measured in meters and t in seconds. Find the velocity and the speed when $t = 5$.

39. $f(t) = 100 + 50t - 4.9t^2$

40. $f(t) = t^{-1} - t$

41. A warm can of soda is placed in a cold refrigerator. Sketch the graph of the temperature of the soda as a function of time. Is the initial rate of change of temperature greater or less than the rate of change after an hour?
42. A roast turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the

temperature is 75°F. The graph shows how the temperature of the turkey decreases and eventually approaches room temperature. By measuring the slope of the tangent, estimate the rate of change of the temperature after an hour.



43. The number N of US cellular phone subscribers (in millions) is shown in the table. (Midyear estimates are given.)

t	1996	1998	2000	2002	2004	2006
N	44	69	109	141	182	233

- (a) Find the average rate of cell phone growth
 (i) from 2002 to 2006 (ii) from 2002 to 2004
 (iii) from 2000 to 2002
 In each case, include the units.
- (b) Estimate the instantaneous rate of growth in 2002 by taking the average of two average rates of change. What are its units?
- (c) Estimate the instantaneous rate of growth in 2002 by measuring the slope of a tangent.
44. The number N of locations of a popular coffeehouse chain is given in the table. (The numbers of locations as of October 1 are given.)

Year	2004	2005	2006	2007	2008
N	8569	10,241	12,440	15,011	16,680

- (a) Find the average rate of growth
 (i) from 2006 to 2008 (ii) from 2006 to 2007
 (iii) from 2005 to 2006
 In each case, include the units.
- (b) Estimate the instantaneous rate of growth in 2006 by taking the average of two average rates of change. What are its units?
- (c) Estimate the instantaneous rate of growth in 2006 by measuring the slope of a tangent.
- (d) Estimate the instantaneous rate of growth in 2007 and compare it with the growth rate in 2006. What do you conclude?
45. The cost (in dollars) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.
- (a) Find the average rate of change of C with respect to x when the production level is changed
 (i) from $x = 100$ to $x = 105$
 (ii) from $x = 100$ to $x = 101$

- (b) Find the instantaneous rate of change of C with respect to x when $x = 100$. (This is called the *marginal cost*. Its significance will be explained in Section 3.7.)

46. If a cylindrical tank holds 100,000 gallons of water, which can be drained from the bottom of the tank in an hour, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V(t) = 100,000\left(1 - \frac{1}{60}t\right)^2 \quad 0 \leq t \leq 60$$

Find the rate at which the water is flowing out of the tank (the instantaneous rate of change of V with respect to t) as a function of t . What are its units? For times $t = 0, 10, 20, 30, 40, 50$, and 60 min, find the flow rate and the amount of water remaining in the tank. Summarize your findings in a sentence or two. At what time is the flow rate the greatest? The least?

47. The cost of producing x ounces of gold from a new gold mine is $C = f(x)$ dollars.

- (a) What is the meaning of the derivative $f'(x)$? What are its units?
- (b) What does the statement $f'(800) = 17$ mean?
- (c) Do you think the values of $f'(x)$ will increase or decrease in the short term? What about the long term? Explain.

48. The number of bacteria after t hours in a controlled laboratory experiment is $n = f(t)$.

- (a) What is the meaning of the derivative $f'(5)$? What are its units?
- (b) Suppose there is an unlimited amount of space and nutrients for the bacteria. Which do you think is larger, $f'(5)$ or $f'(10)$? If the supply of nutrients is limited, would that affect your conclusion? Explain.

49. Let $T(t)$ be the temperature (in °F) in Phoenix t hours after midnight on September 10, 2008. The table shows values of this function recorded every two hours. What is the meaning of $T'(8)$? Estimate its value.

t	0	2	4	6	8	10	12	14
T	82	75	74	75	84	90	93	94

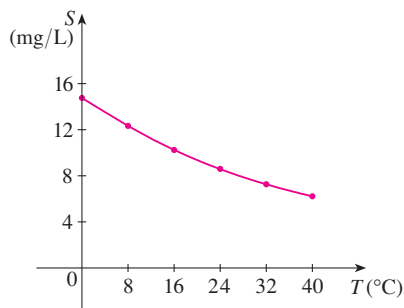
50. The quantity (in pounds) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per pound is $Q = f(p)$.

- (a) What is the meaning of the derivative $f'(8)$? What are its units?
- (b) Is $f'(8)$ positive or negative? Explain.

51. The quantity of oxygen that can dissolve in water depends on the temperature of the water. (So thermal pollution influences

the oxygen content of water.) The graph shows how oxygen solubility S varies as a function of the water temperature T .

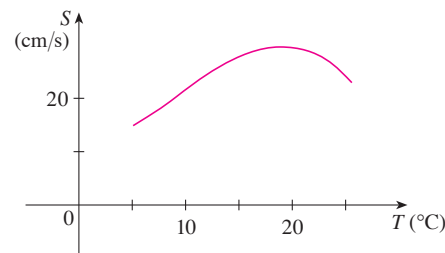
- (a) What is the meaning of the derivative $S'(T)$? What are its units?
 (b) Estimate the value of $S'(16)$ and interpret it.



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52. The graph shows the influence of the temperature T on the maximum sustainable swimming speed S of Coho salmon.
 (a) What is the meaning of the derivative $S'(T)$? What are its units?

- (b) Estimate the values of $S'(15)$ and $S'(25)$ and interpret them.



- 53–54 Determine whether $f'(0)$ exists.

53.
$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

54.
$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

WRITING PROJECT

EARLY METHODS FOR FINDING TANGENTS

The first person to formulate explicitly the ideas of limits and derivatives was Sir Isaac Newton in the 1660s. But Newton acknowledged that “If I have seen further than other men, it is because I have stood on the shoulders of giants.” Two of those giants were Pierre Fermat (1601–1665) and Newton’s mentor at Cambridge, Isaac Barrow (1630–1677). Newton was familiar with the methods that these men used to find tangent lines, and their methods played a role in Newton’s eventual formulation of calculus.

The following references contain explanations of these methods. Read one or more of the references and write a report comparing the methods of either Fermat or Barrow to modern methods. In particular, use the method of Section 2.7 to find an equation of the tangent line to the curve $y = x^3 + 2x$ at the point $(1, 3)$ and show how either Fermat or Barrow would have solved the same problem. Although you used derivatives and they did not, point out similarities between the methods.

1. Carl Boyer and Uta Merzbach, *A History of Mathematics* (New York: Wiley, 1989), pp. 389, 432.
2. C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), pp. 124, 132.
3. Howard Eves, *An Introduction to the History of Mathematics*, 6th ed. (New York: Saunders, 1990), pp. 391, 395.
4. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), pp. 344, 346.

2.8 The Derivative as a Function

In the preceding section we considered the derivative of a function f at a fixed number a :

$$\boxed{1} \quad f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

Here we change our point of view and let the number a vary. If we replace a in Equation 1 by a variable x , we obtain

$$\boxed{2} \quad f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

Given any number x for which this limit exists, we assign to x the number $f'(x)$. So we can regard f' as a new function, called the **derivative of f** and defined by Equation 2. We know that the value of f' at x , $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

The function f' is called the derivative of f because it has been “derived” from f by the limiting operation in Equation 2. The domain of f' is the set $\{x \mid f'(x) \text{ exists}\}$ and may be smaller than the domain of f .

V EXAMPLE 1 The graph of a function f is given in Figure 1. Use it to sketch the graph of the derivative f' .

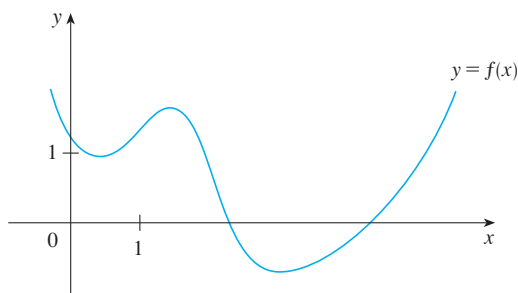
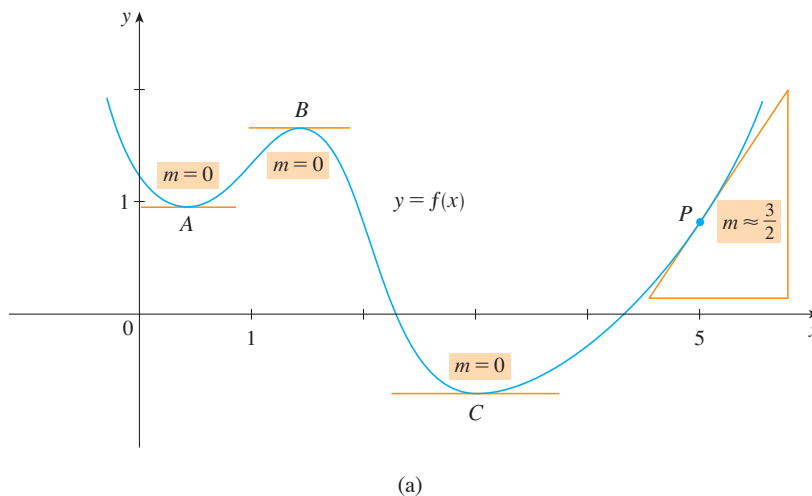


FIGURE 1

SOLUTION We can estimate the value of the derivative at any value of x by drawing the tangent at the point $(x, f(x))$ and estimating its slope. For instance, for $x = 5$ we draw the tangent at P in Figure 2(a) and estimate its slope to be about $\frac{3}{2}$, so $f'(5) \approx 1.5$. This allows us to plot the point $P'(5, 1.5)$ on the graph of f' directly beneath P . Repeating this procedure at several points, we get the graph shown in Figure 2(b). Notice that the tangents at A , B , and C are horizontal, so the derivative is 0 there and the graph of f' crosses the x -axis at the points A' , B' , and C' , directly beneath A , B , and C . Between A and B the tangents have positive slope, so $f'(x)$ is positive there. But between B and C the tangents have negative slope, so $f'(x)$ is negative there.



TEC Visual 2.8 shows an animation of Figure 2 for several functions.

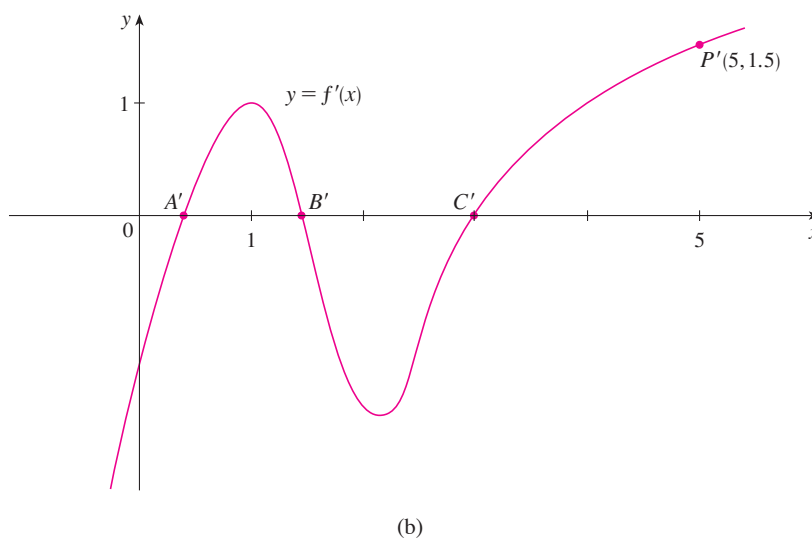


FIGURE 2

EXAMPLE 2

- (a) If $f(x) = x^3 - x$, find a formula for $f'(x)$.
 (b) Illustrate by comparing the graphs of f and f' .

SOLUTION

(a) When using Equation 2 to compute a derivative, we must remember that the variable is h and that x is temporarily regarded as a constant during the calculation of the limit.

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} = \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 - 1) = 3x^2 - 1
 \end{aligned}$$

(b) We use a graphing device to graph f and f' in Figure 3. Notice that $f'(x) = 0$ when f has horizontal tangents and $f'(x)$ is positive when the tangents have positive slope. So these graphs serve as a check on our work in part (a).

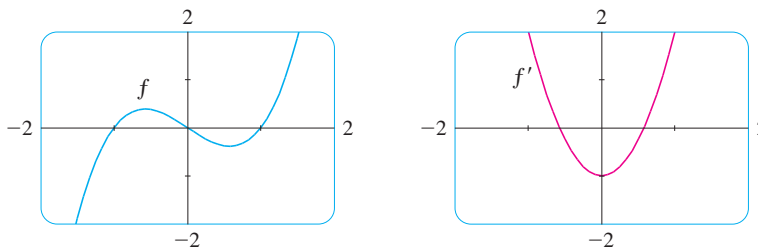


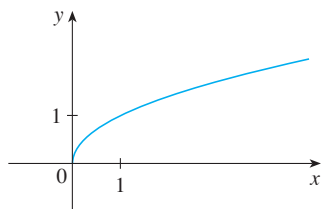
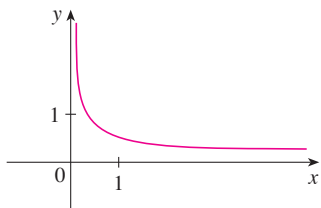
FIGURE 3

EXAMPLE 3 If $f(x) = \sqrt{x}$, find the derivative of f . State the domain of f' .

SOLUTION

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\ &= \lim_{h \rightarrow 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

Here we rationalize the numerator.

(a) $f(x) = \sqrt{x}$ (b) $f'(x) = \frac{1}{2\sqrt{x}}$

We see that $f'(x)$ exists if $x > 0$, so the domain of f' is $(0, \infty)$. This is smaller than the domain of f , which is $[0, \infty)$.

Let's check to see that the result of Example 3 is reasonable by looking at the graphs of f and f' in Figure 4. When x is close to 0, \sqrt{x} is also close to 0, so $f'(x) = 1/(2\sqrt{x})$ is very large and this corresponds to the steep tangent lines near $(0, 0)$ in Figure 4(a) and the large values of $f'(x)$ just to the right of 0 in Figure 4(b). When x is large, $f'(x)$ is very small and this corresponds to the flatter tangent lines at the far right of the graph of f and the horizontal asymptote of the graph of f' .

EXAMPLE 4 Find f' if $f(x) = \frac{1-x}{2+x}$.

SOLUTION

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1-(x+h)}{2+(x+h)} - \frac{1-x}{2+x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1-x-h)(2+x) - (1-x)(2+x+h)}{h(2+x+h)(2+x)} \\ &= \lim_{h \rightarrow 0} \frac{(2-x-2h-x^2-xh) - (2-x+h-x^2-xh)}{h(2+x+h)(2+x)} \\ &= \lim_{h \rightarrow 0} \frac{-3h}{h(2+x+h)(2+x)} = \lim_{h \rightarrow 0} \frac{-3}{(2+x+h)(2+x)} = -\frac{3}{(2+x)^2} \end{aligned}$$

$$\frac{\frac{a}{b} - \frac{c}{d}}{e} = \frac{ad - bc}{bd} \cdot \frac{1}{e}$$

FIGURE 4

Other Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y , then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols D and d/dx are called **differentiation operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative.

The symbol dy/dx , which was introduced by Leibniz, should not be regarded as a ratio (for the time being); it is simply a synonym for $f'(x)$. Nonetheless, it is a very useful and suggestive notation, especially when used in conjunction with increment notation. Referring to Equation 2.7.6, we can rewrite the definition of derivative in Leibniz notation in the form

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

If we want to indicate the value of a derivative dy/dx in Leibniz notation at a specific number a , we use the notation

$$\left. \frac{dy}{dx} \right|_{x=a} \quad \text{or} \quad \left. \frac{dy}{dx} \right]_{x=a}$$

which is a synonym for $f'(a)$.

3 Definition A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval (a, b)** [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

V EXAMPLE 5 Where is the function $f(x) = |x|$ differentiable?

SOLUTION If $x > 0$, then $|x| = x$ and we can choose h small enough that $x + h > 0$ and hence $|x + h| = x + h$. Therefore, for $x > 0$, we have

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{|x + h| - |x|}{h} = \lim_{h \rightarrow 0} \frac{(x + h) - x}{h} \\ &= \lim_{h \rightarrow 0} \frac{h}{h} = \lim_{h \rightarrow 0} 1 = 1 \end{aligned}$$

and so f is differentiable for any $x > 0$.

Similarly, for $x < 0$ we have $|x| = -x$ and h can be chosen small enough that $x + h < 0$ and so $|x + h| = -(x + h)$. Therefore, for $x < 0$,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{|x + h| - |x|}{h} = \lim_{h \rightarrow 0} \frac{-(x + h) - (-x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{h} = \lim_{h \rightarrow 0} (-1) = -1 \end{aligned}$$

and so f is differentiable for any $x < 0$.

Leibniz

Gottfried Wilhelm Leibniz was born in Leipzig in 1646 and studied law, theology, philosophy, and mathematics at the university there, graduating with a bachelor's degree at age 17. After earning his doctorate in law at age 20, Leibniz entered the diplomatic service and spent most of his life traveling to the capitals of Europe on political missions. In particular, he worked to avert a French military threat against Germany and attempted to reconcile the Catholic and Protestant churches.

His serious study of mathematics did not begin until 1672 while he was on a diplomatic mission in Paris. There he built a calculating machine and met scientists, like Huygens, who directed his attention to the latest developments in mathematics and science. Leibniz sought to develop a symbolic logic and system of notation that would simplify logical reasoning. In particular, the version of calculus that he published in 1684 established the notation and the rules for finding derivatives that we use today.

Unfortunately, a dreadful priority dispute arose in the 1690s between the followers of Newton and those of Leibniz as to who had invented calculus first. Leibniz was even accused of plagiarism by members of the Royal Society in England. The truth is that each man invented calculus independently. Newton arrived at his version of calculus first but, because of his fear of controversy, did not publish it immediately. So Leibniz's 1684 account of calculus was the first to be published.

For $x = 0$ we have to investigate

$$\begin{aligned} f'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{|0+h| - |0|}{h} \quad (\text{if it exists}) \end{aligned}$$

Let's compute the left and right limits separately:

$$\lim_{h \rightarrow 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \rightarrow 0^+} \frac{|h|}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = \lim_{h \rightarrow 0^+} 1 = 1$$

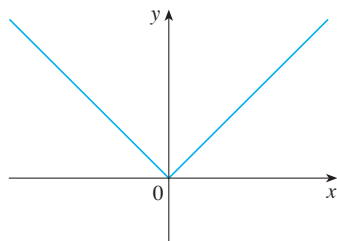
$$\text{and} \quad \lim_{h \rightarrow 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \rightarrow 0^-} \frac{|h|}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = \lim_{h \rightarrow 0^-} (-1) = -1$$

Since these limits are different, $f'(0)$ does not exist. Thus f is differentiable at all x except 0.

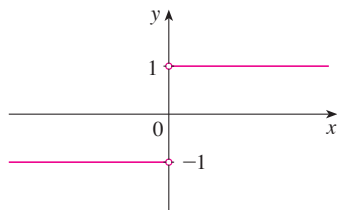
A formula for f' is given by

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and its graph is shown in Figure 5(b). The fact that $f'(0)$ does not exist is reflected geometrically in the fact that the curve $y = |x|$ does not have a tangent line at $(0, 0)$. [See Figure 5(a).]



(a) $y = f(x) = |x|$



(b) $y = f'(x)$

FIGURE 5

Both continuity and differentiability are desirable properties for a function to have. The following theorem shows how these properties are related.

4 Theorem If f is differentiable at a , then f is continuous at a .

PROOF To prove that f is continuous at a , we have to show that $\lim_{x \rightarrow a} f(x) = f(a)$. We do this by showing that the difference $f(x) - f(a)$ approaches 0.

The given information is that f is differentiable at a , that is,

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

exists (see Equation 2.7.5). To connect the given and the unknown, we divide and multiply $f(x) - f(a)$ by $x - a$ (which we can do when $x \neq a$):

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a} (x - a)$$

Thus, using the Product Law and (2.7.5), we can write


$$\begin{aligned} \lim_{x \rightarrow a} [f(x) - f(a)] &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} (x - a) \\ &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \rightarrow a} (x - a) \\ &= f'(a) \cdot 0 = 0 \end{aligned}$$

PS An important aspect of problem solving is trying to find a connection between the given and the unknown. See Step 2 (Think of a Plan) in *Principles of Problem Solving* on page 75.

To use what we have just proved, we start with $f(x)$ and add and subtract $f(a)$:

$$\begin{aligned}\lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} [f(a) + (f(x) - f(a))] \\ &= \lim_{x \rightarrow a} f(a) + \lim_{x \rightarrow a} [f(x) - f(a)] \\ &= f(a) + 0 = f(a)\end{aligned}$$

Therefore f is continuous at a .

 **NOTE** The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable. For instance, the function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} |x| = 0 = f(0)$$

(See Example 7 in Section 2.3.) But in Example 5 we showed that f is not differentiable at 0.

How Can a Function Fail to Be Differentiable?

We saw that the function $y = |x|$ in Example 5 is not differentiable at 0 and Figure 5(a) shows that its graph changes direction abruptly when $x = 0$. In general, if the graph of a function f has a “corner” or “kink” in it, then the graph of f has no tangent at this point and f is not differentiable there. [In trying to compute $f'(a)$, we find that the left and right limits are different.]

Theorem 4 gives another way for a function not to have a derivative. It says that if f is not continuous at a , then f is not differentiable at a . So at any discontinuity (for instance, a jump discontinuity) f fails to be differentiable.

A third possibility is that the curve has a **vertical tangent line** when $x = a$; that is, f is continuous at a and

$$\lim_{x \rightarrow a} |f'(x)| = \infty$$

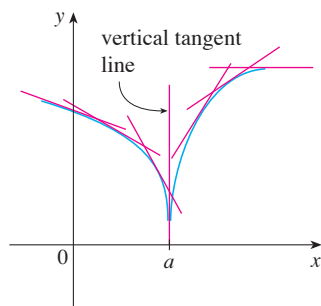


FIGURE 6

This means that the tangent lines become steeper and steeper as $x \rightarrow a$. Figure 6 shows one way that this can happen; Figure 7(c) shows another. Figure 7 illustrates the three possibilities that we have discussed.

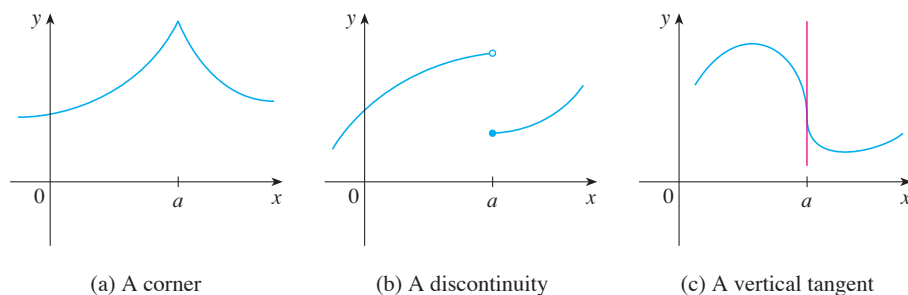


FIGURE 7

Three ways for f not to be differentiable at a

A graphing calculator or computer provides another way of looking at differentiability. If f is differentiable at a , then when we zoom in toward the point $(a, f(a))$ the graph

straightens out and appears more and more like a line. (See Figure 8. We saw a specific example of this in Figure 2 in Section 2.7.) But no matter how much we zoom in toward a point like the ones in Figures 6 and 7(a), we can't eliminate the sharp point or corner (see Figure 9).

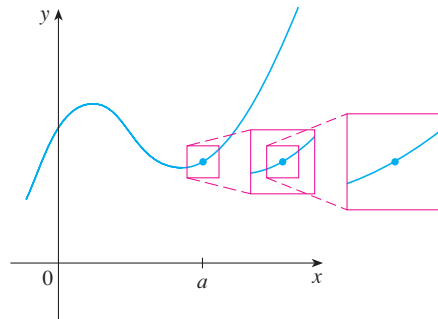


FIGURE 8
 f is differentiable at a .

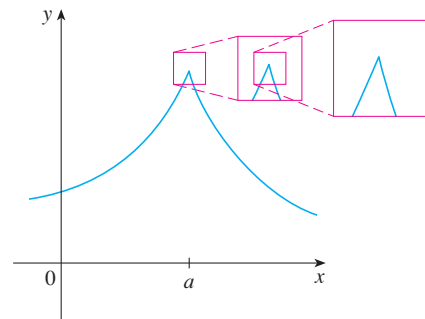


FIGURE 9
 f is not differentiable at a .

Higher Derivatives

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f because it is the derivative of the derivative of f . Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2}$$

EXAMPLE 6 If $f(x) = x^3 - x$, find and interpret $f''(x)$.

SOLUTION In Example 2 we found that the first derivative is $f'(x) = 3x^2 - 1$. So the second derivative is

$$\begin{aligned} f''(x) &= (f')'(x) = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h} \\ &= \lim_{h \rightarrow 0} (6x + 3h) = 6x \end{aligned}$$

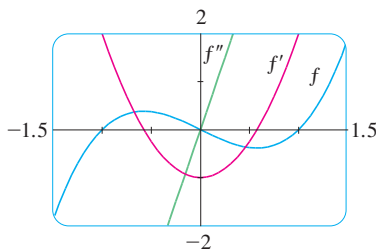


FIGURE 10

TEC In Module 2.8 you can see how changing the coefficients of a polynomial f affects the appearance of the graphs of f , f' , and f'' .

The graphs of f , f' , and f'' are shown in Figure 10.

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$. In other words, it is the rate of change of the slope of the original curve $y = f(x)$.

Notice from Figure 10 that $f''(x)$ is negative when $y = f'(x)$ has negative slope and positive when $y = f'(x)$ has positive slope. So the graphs serve as a check on our calculations.

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is *acceleration*, which we define as follows.

If $s = s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $v(t)$ of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t)$$

or, in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

The **third derivative** f''' is the derivative of the second derivative: $f''' = (f'')'$. So $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$. If $y = f(x)$, then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{d}{dx} \left(\frac{d^2y}{dx^2} \right) = \frac{d^3y}{dx^3}$$

The process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$. In general, the n th derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times. If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

EXAMPLE 7 If $f(x) = x^3 - x$, find $f'''(x)$ and $f^{(4)}(x)$.

SOLUTION In Example 6 we found that $f''(x) = 6x$. The graph of the second derivative has equation $y = 6x$ and so it is a straight line with slope 6. Since the derivative $f'''(x)$ is the slope of $f''(x)$, we have

$$f'''(x) = 6$$

for all values of x . So f''' is a constant function and its graph is a horizontal line. Therefore, for all values of x ,

$$f^{(4)}(x) = 0$$

We can also interpret the third derivative physically in the case where the function is the position function $s = s(t)$ of an object that moves along a straight line. Because $s''' = (s'')' = a'$, the third derivative of the position function is the derivative of the acceleration function and is called the **jerk**:

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

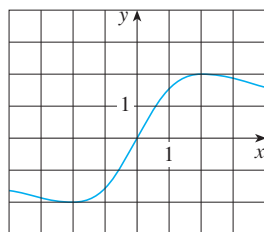
Thus the jerk j is the rate of change of acceleration. It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.

We have seen that one application of second and third derivatives occurs in analyzing the motion of objects using acceleration and jerk. We will investigate another application of second derivatives in Section 4.3, where we show how knowledge of f'' gives us information about the shape of the graph of f . In Chapter 11 we will see how second and higher derivatives enable us to represent functions as sums of infinite series.

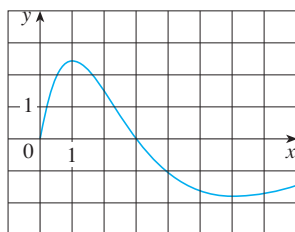
2.8 Exercises

1–2 Use the given graph to estimate the value of each derivative. Then sketch the graph of f' .

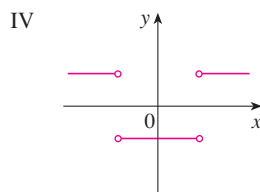
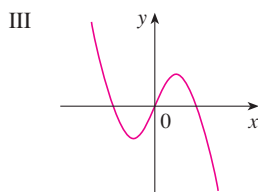
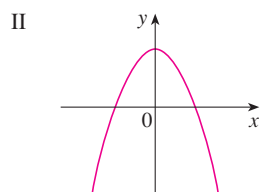
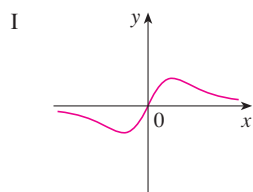
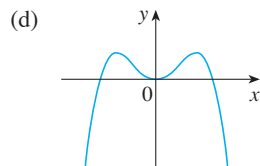
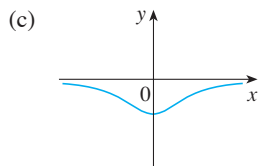
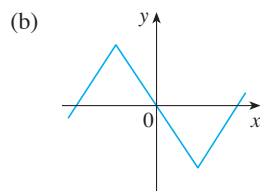
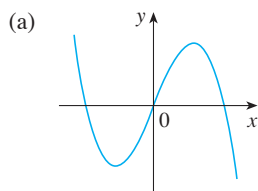
1. (a) $f'(-3)$
 (b) $f'(-2)$
 (c) $f'(-1)$
 (d) $f'(0)$
 (e) $f'(1)$
 (f) $f'(2)$
 (g) $f'(3)$



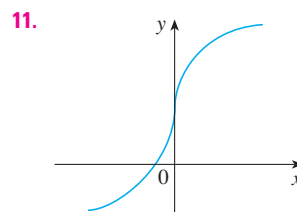
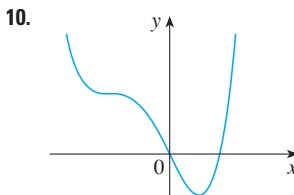
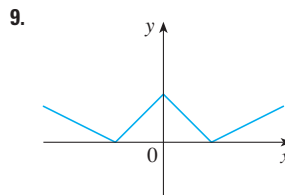
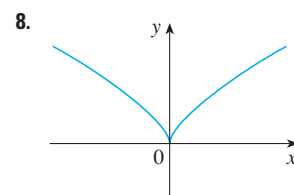
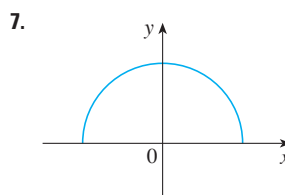
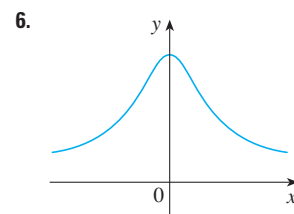
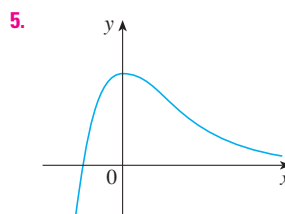
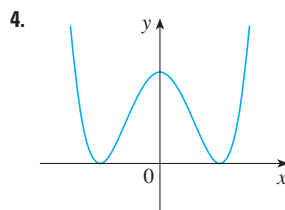
2. (a) $f'(0)$
 (b) $f'(1)$
 (c) $f'(2)$
 (d) $f'(3)$
 (e) $f'(4)$
 (f) $f'(5)$
 (g) $f'(6)$
 (h) $f'(7)$



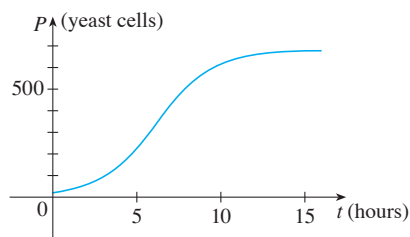
3. Match the graph of each function in (a)–(d) with the graph of its derivative in I–IV. Give reasons for your choices.



4–11 Trace or copy the graph of the given function f . (Assume that the axes have equal scales.) Then use the method of Example 1 to sketch the graph of f' below it.

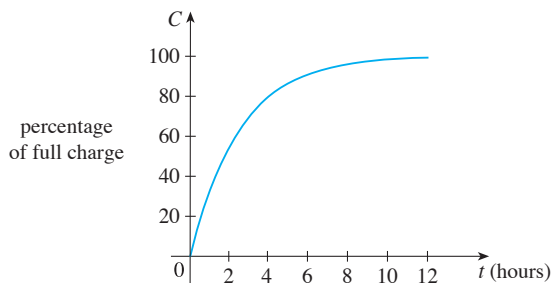


12. Shown is the graph of the population function $P(t)$ for yeast cells in a laboratory culture. Use the method of Example 1 to

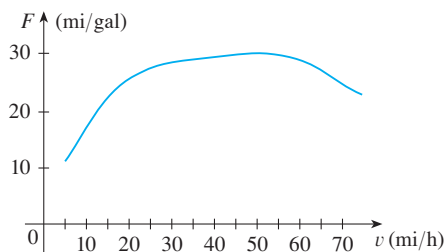


graph the derivative $P'(t)$. What does the graph of P' tell us about the yeast population?

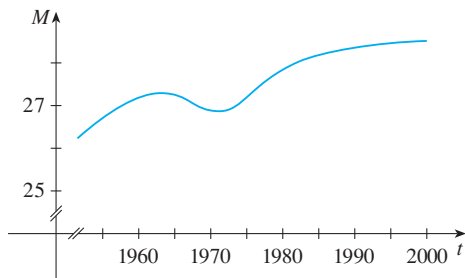
13. A rechargeable battery is plugged into a charger. The graph shows $C(t)$, the percentage of full capacity that the battery reaches as a function of time t elapsed (in hours).
- What is the meaning of the derivative $C'(t)$?
 - Sketch the graph of $C'(t)$. What does the graph tell you?



14. The graph (from the US Department of Energy) shows how driving speed affects gas mileage. Fuel economy F is measured in miles per gallon and speed v is measured in miles per hour.
- What is the meaning of the derivative $F'(v)$?
 - Sketch the graph of $F'(v)$.
 - At what speed should you drive if you want to save on gas?



15. The graph shows how the average age of first marriage of Japanese men varied in the last half of the 20th century. Sketch the graph of the derivative function $M'(t)$. During which years was the derivative negative?



16–18 Make a careful sketch of the graph of f and below it sketch the graph of f' in the same manner as in Exercises 4–11. Can you guess a formula for $f'(x)$ from its graph?

16. $f(x) = \sin x$

17. $f(x) = e^x$

18. $f(x) = \ln x$

19. Let $f(x) = x^2$.

- Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, and $f'(2)$ by using a graphing device to zoom in on the graph of f .
- Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, and $f'(-2)$.
- Use the results from parts (a) and (b) to guess a formula for $f'(x)$.
- Use the definition of derivative to prove that your guess in part (c) is correct.

20. Let $f(x) = x^3$.

- Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, $f'(2)$, and $f'(3)$ by using a graphing device to zoom in on the graph of f .
- Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, $f'(-2)$, and $f'(-3)$.
- Use the values from parts (a) and (b) to graph f' .
- Guess a formula for $f'(x)$.
- Use the definition of derivative to prove that your guess in part (d) is correct.

21–31 Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

21. $f(x) = \frac{1}{2}x - \frac{1}{3}$

22. $f(x) = mx + b$

23. $f(t) = 5t - 9t^2$

24. $f(x) = 1.5x^2 - x + 3.7$

25. $f(x) = x^2 - 2x^3$

26. $g(t) = \frac{1}{\sqrt{t}}$

27. $g(x) = \sqrt{9 - x}$

28. $f(x) = \frac{x^2 - 1}{2x - 3}$

29. $G(t) = \frac{1 - 2t}{3 + t}$

30. $f(x) = x^{3/2}$

31. $f(x) = x^4$

32. (a) Sketch the graph of $f(x) = \sqrt{6 - x}$ by starting with the graph of $y = \sqrt{x}$ and using the transformations of Section 1.3.

- Use the graph from part (a) to sketch the graph of f' .
- Use the definition of a derivative to find $f'(x)$. What are the domains of f and f' ?

(d) Use a graphing device to graph f' and compare with your sketch in part (b).

33. (a) If $f(x) = x^4 + 2x$, find $f'(x)$.

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f' .

34. (a) If $f(x) = x + 1/x$, find $f'(x)$.

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f' .

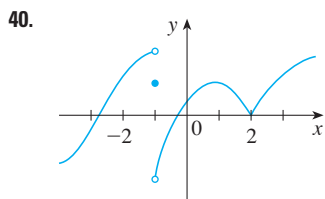
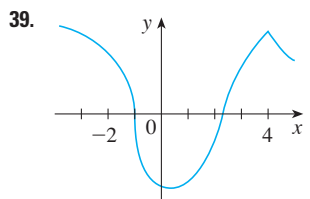
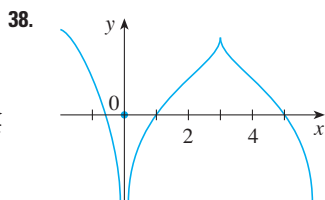
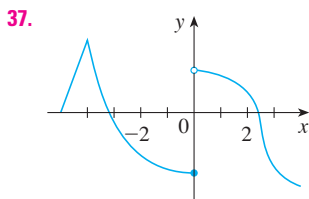
35. The unemployment rate $U(t)$ varies with time. The table (from the Bureau of Labor Statistics) gives the percentage of unemployed in the US labor force from 1999 to 2008.

t	$U(t)$	t	$U(t)$
1999	4.2	2004	5.5
2000	4.0	2005	5.1
2001	4.7	2006	4.6
2002	5.8	2007	4.6
2003	6.0	2008	5.8

- (a) What is the meaning of $U'(t)$? What are its units?
 (b) Construct a table of estimated values for $U'(t)$.
36. Let $P(t)$ be the percentage of Americans under the age of 18 at time t . The table gives values of this function in census years from 1950 to 2000.

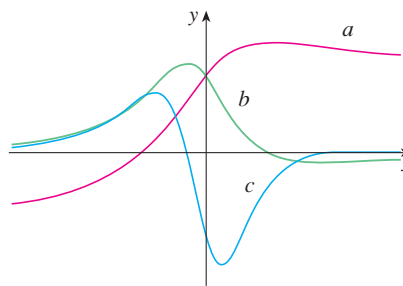
t	$P(t)$	t	$P(t)$
1950	31.1	1980	28.0
1960	35.7	1990	25.7
1970	34.0	2000	25.7

- (a) What is the meaning of $P'(t)$? What are its units?
 (b) Construct a table of estimated values for $P'(t)$.
 (c) Graph P and P' .
 (d) How would it be possible to get more accurate values for $P'(t)$?
- 37–40 The graph of f is given. State, with reasons, the numbers at which f is not differentiable.

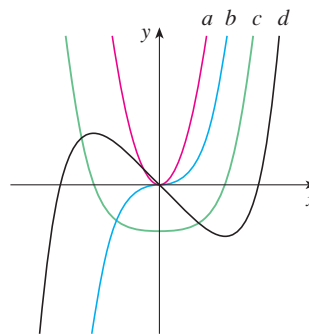


41. Graph the function $f(x) = x + \sqrt{|x|}$. Zoom in repeatedly, first toward the point $(-1, 0)$ and then toward the origin. What is different about the behavior of f in the vicinity of these two points? What do you conclude about the differentiability of f ?
42. Zoom in toward the points $(1, 0)$, $(0, 1)$, and $(-1, 0)$ on the graph of the function $g(x) = (x^2 - 1)^{2/3}$. What do you notice? Account for what you see in terms of the differentiability of g .

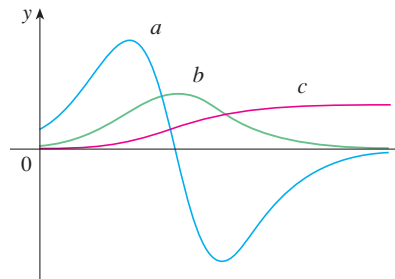
43. The figure shows the graphs of f , f' , and f'' . Identify each curve, and explain your choices.



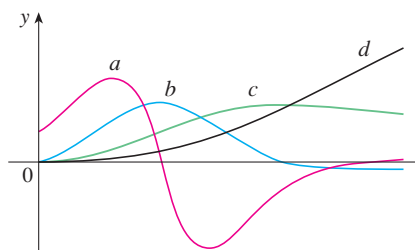
44. The figure shows graphs of f , f' , f'' , and f''' . Identify each curve, and explain your choices.




45. The figure shows the graphs of three functions. One is the position function of a car, one is the velocity of the car, and one is its acceleration. Identify each curve, and explain your choices.




46. The figure shows the graphs of four functions. One is the position function of a car, one is the velocity of the car, one is its acceleration, and one is its jerk. Identify each curve, and explain your choices.



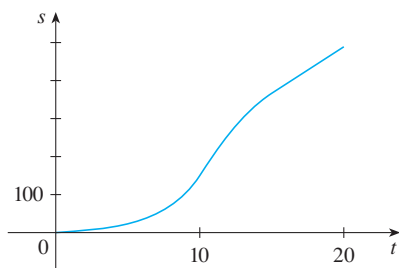
-  **47–48** Use the definition of a derivative to find $f'(x)$ and $f''(x)$. Then graph f , f' , and f'' on a common screen and check to see if your answers are reasonable.


47. $f(x) = 3x^2 + 2x + 1$

48. $f(x) = x^3 - 3x$

-  **49.** If $f(x) = 2x^2 - x^3$, find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$. Graph f , f' , f'' , and f''' on a common screen. Are the graphs consistent with the geometric interpretations of these derivatives?

- 50.** (a) The graph of a position function of a car is shown, where s is measured in feet and t in seconds. Use it to graph the velocity and acceleration of the car. What is the acceleration at $t = 10$ seconds?



- (b) Use the acceleration curve from part (a) to estimate the jerk at $t = 10$ seconds. What are the units for jerk?
- 51.** Let $f(x) = \sqrt[3]{x}$.
- (a) If $a \neq 0$, use Equation 2.7.5 to find $f'(a)$.
- (b) Show that $f'(0)$ does not exist.
- (c) Show that $y = \sqrt[3]{x}$ has a vertical tangent line at $(0, 0)$. (Recall the shape of the graph of f . See Figure 13 in Section 1.2.)
- 52.** (a) If $g(x) = x^{2/3}$, show that $g'(0)$ does not exist.
- (b) If $a \neq 0$, find $g'(a)$.
- (c) Show that $y = x^{2/3}$ has a vertical tangent line at $(0, 0)$.
-  (d) Illustrate part (c) by graphing $y = x^{2/3}$.

- 53.** Show that the function $f(x) = |x - 6|$ is not differentiable at 6. Find a formula for f' and sketch its graph.

- 54.** Where is the greatest integer function $f(x) = \llbracket x \rrbracket$ not differentiable? Find a formula for f' and sketch its graph.

- 55.** (a) Sketch the graph of the function $f(x) = x|x|$.
- (b) For what values of x is f differentiable?
- (c) Find a formula for f' .

- 56.** The **left-hand** and **right-hand derivatives** of f at a are defined by

$$f'_-(a) = \lim_{h \rightarrow 0^-} \frac{f(a+h) - f(a)}{h}$$

and
$$f'_+(a) = \lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h}$$

if these limits exist. Then $f'(a)$ exists if and only if these one-sided derivatives exist and are equal.

- (a) Find $f'_-(4)$ and $f'_+(4)$ for the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 5 - x & \text{if } 0 < x < 4 \\ \frac{1}{5 - x} & \text{if } x \geq 4 \end{cases}$$

- (b) Sketch the graph of f .
- (c) Where is f discontinuous?
- (d) Where is f not differentiable?

- 57.** Recall that a function f is called *even* if $f(-x) = f(x)$ for all x in its domain and *odd* if $f(-x) = -f(x)$ for all such x . Prove each of the following.

- (a) The derivative of an even function is an odd function.
- (b) The derivative of an odd function is an even function.

- 58.** When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running.

- (a) Sketch a possible graph of T as a function of the time t that has elapsed since the faucet was turned on.
- (b) Describe how the rate of change of T with respect to t varies as t increases.
- (c) Sketch a graph of the derivative of T .

- 59.** Let ℓ be the tangent line to the parabola $y = x^2$ at the point $(1, 1)$. The *angle of inclination* of ℓ is the angle ϕ that ℓ makes with the positive direction of the x -axis. Calculate ϕ correct to the nearest degree.

2 Review

Concept Check

- 1.** Explain what each of the following means and illustrate with a sketch.

(a) $\lim_{x \rightarrow a} f(x) = L$

(b) $\lim_{x \rightarrow a^+} f(x) = L$

(c) $\lim_{x \rightarrow a^-} f(x) = L$

(d) $\lim_{x \rightarrow a} f(x) = \infty$

(e) $\lim_{x \rightarrow \infty} f(x) = L$

- 2.** Describe several ways in which a limit can fail to exist. Illustrate with sketches.

- 3.** State the following Limit Laws.

(a) Sum Law

(b) Difference Law

(c) Constant Multiple Law

(d) Product Law

(e) Quotient Law

(f) Power Law

(g) Root Law

4. What does the Squeeze Theorem say?
5. (a) What does it mean to say that the line $x = a$ is a vertical asymptote of the curve $y = f(x)$? Draw curves to illustrate the various possibilities.
(b) What does it mean to say that the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$? Draw curves to illustrate the various possibilities.
6. Which of the following curves have vertical asymptotes? Which have horizontal asymptotes?

(a) $y = x^4$	(b) $y = \sin x$
(c) $y = \tan x$	(d) $y = \tan^{-1}x$
(e) $y = e^x$	(f) $y = \ln x$
(g) $y = 1/x$	(h) $y = \sqrt{x}$
7. (a) What does it mean for f to be continuous at a ?
(b) What does it mean for f to be continuous on the interval $(-\infty, \infty)$? What can you say about the graph of such a function?
8. What does the Intermediate Value Theorem say?
9. Write an expression for the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.
10. Suppose an object moves along a straight line with position $f(t)$ at time t . Write an expression for the instantaneous velocity of the object at time $t = a$. How can you interpret this velocity in terms of the graph of f ?
11. If $y = f(x)$ and x changes from x_1 to x_2 , write expressions for the following.
 - (a) The average rate of change of y with respect to x over the interval $[x_1, x_2]$.
 - (b) The instantaneous rate of change of y with respect to x at $x = x_1$.
12. Define the derivative $f'(a)$. Discuss two ways of interpreting this number.
13. Define the second derivative of f . If $f(t)$ is the position function of a particle, how can you interpret the second derivative?
14. (a) What does it mean for f to be differentiable at a ?
(b) What is the relation between the differentiability and continuity of a function?
(c) Sketch the graph of a function that is continuous but not differentiable at $a = 2$.
15. Describe several ways in which a function can fail to be differentiable. Illustrate with sketches.

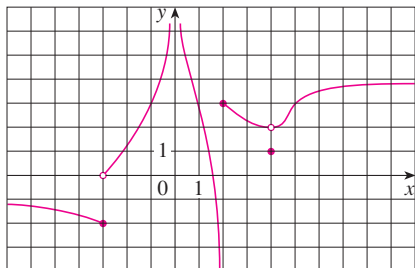
True-False Quiz

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. $\lim_{x \rightarrow 4} \left(\frac{2x}{x-4} - \frac{8}{x-4} \right) = \lim_{x \rightarrow 4} \frac{2x}{x-4} - \lim_{x \rightarrow 4} \frac{8}{x-4}$
2. $\lim_{x \rightarrow 1} \frac{x^2 + 6x - 7}{x^2 + 5x - 6} = \frac{\lim_{x \rightarrow 1} (x^2 + 6x - 7)}{\lim_{x \rightarrow 1} (x^2 + 5x - 6)}$
3. $\lim_{x \rightarrow 1} \frac{x-3}{x^2 + 2x - 4} = \frac{\lim_{x \rightarrow 1} (x-3)}{\lim_{x \rightarrow 1} (x^2 + 2x - 4)}$
4. If $\lim_{x \rightarrow 5} f(x) = 2$ and $\lim_{x \rightarrow 5} g(x) = 0$, then $\lim_{x \rightarrow 5} [f(x)/g(x)]$ does not exist.
5. If $\lim_{x \rightarrow 5} f(x) = 0$ and $\lim_{x \rightarrow 5} g(x) = 0$, then $\lim_{x \rightarrow 5} [f(x)/g(x)]$ does not exist.
6. If neither $\lim_{x \rightarrow a} f(x)$ nor $\lim_{x \rightarrow a} g(x)$ exists, then $\lim_{x \rightarrow a} [f(x) + g(x)]$ does not exist.
7. If $\lim_{x \rightarrow a} f(x)$ exists but $\lim_{x \rightarrow a} g(x)$ does not exist, then $\lim_{x \rightarrow a} [f(x) + g(x)]$ does not exist.
8. If $\lim_{x \rightarrow 6} [f(x)g(x)]$ exists, then the limit must be $f(6)g(6)$.
9. If p is a polynomial, then $\lim_{x \rightarrow b} p(x) = p(b)$.
10. If $\lim_{x \rightarrow 0} f(x) = \infty$ and $\lim_{x \rightarrow 0} g(x) = \infty$, then $\lim_{x \rightarrow 0} [f(x) - g(x)] = 0$.
11. A function can have two different horizontal asymptotes.
12. If f has domain $[0, \infty)$ and has no horizontal asymptote, then $\lim_{x \rightarrow \infty} f(x) = \infty$ or $\lim_{x \rightarrow \infty} f(x) = -\infty$.
13. If the line $x = 1$ is a vertical asymptote of $y = f(x)$, then f is not defined at 1.
14. If $f(1) > 0$ and $f(3) < 0$, then there exists a number c between 1 and 3 such that $f(c) = 0$.
15. If f is continuous at 5 and $f(5) = 2$ and $f(4) = 3$, then $\lim_{x \rightarrow 2} f(4x^2 - 11) = 2$.
16. If f is continuous on $[-1, 1]$ and $f(-1) = 4$ and $f(1) = 3$, then there exists a number r such that $|r| < 1$ and $f(r) = \pi$.
17. Let f be a function such that $\lim_{x \rightarrow 0} f(x) = 6$. Then there exists a number δ such that if $0 < |x| < \delta$, then $|f(x) - 6| < 1$.
18. If $f(x) > 1$ for all x and $\lim_{x \rightarrow 0} f(x)$ exists, then $\lim_{x \rightarrow 0} f(x) > 1$.
19. If f is continuous at a , then f is differentiable at a .
20. If $f'(r)$ exists, then $\lim_{x \rightarrow r} f(x) = f(r)$.
21. $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx} \right)^2$
22. The equation $x^{10} - 10x^2 + 5 = 0$ has a root in the interval $(0, 2)$.
23. If f is continuous at a , so is $|f|$.
24. If $|f|$ is continuous at a , so is f .

Exercises

1. The graph of f is given.
- (a) Find each limit, or explain why it does not exist.
- (i) $\lim_{x \rightarrow 2^+} f(x)$ (ii) $\lim_{x \rightarrow -3^+} f(x)$
- (iii) $\lim_{x \rightarrow -3} f(x)$ (iv) $\lim_{x \rightarrow 4} f(x)$
- (v) $\lim_{x \rightarrow 0} f(x)$ (vi) $\lim_{x \rightarrow 2^-} f(x)$
- (vii) $\lim_{x \rightarrow \infty} f(x)$ (viii) $\lim_{x \rightarrow -\infty} f(x)$
- (b) State the equations of the horizontal asymptotes.
- (c) State the equations of the vertical asymptotes.
- (d) At what numbers is f discontinuous? Explain.



2. Sketch the graph of an example of a function f that satisfies all of the following conditions:

$$\lim_{x \rightarrow -\infty} f(x) = -2, \quad \lim_{x \rightarrow \infty} f(x) = 0, \quad \lim_{x \rightarrow -3} f(x) = \infty,$$

$$\lim_{x \rightarrow 3^-} f(x) = -\infty, \quad \lim_{x \rightarrow 3^+} f(x) = 2,$$

f is continuous from the right at 3

- 3–20 Find the limit.

3. $\lim_{x \rightarrow 1} e^{x^3 - x}$

5. $\lim_{x \rightarrow -3} \frac{x^2 - 9}{x^2 + 2x - 3}$

7. $\lim_{h \rightarrow 0} \frac{(h-1)^3 + 1}{h}$

9. $\lim_{r \rightarrow 9} \frac{\sqrt{r}}{(r-9)^4}$

11. $\lim_{u \rightarrow 1} \frac{u^4 - 1}{u^3 + 5u^2 - 6u}$

13. $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 - 9}}{2x - 6}$

15. $\lim_{x \rightarrow \pi^-} \ln(\sin x)$

4. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x^2 + 2x - 3}$

6. $\lim_{x \rightarrow 1^+} \frac{x^2 - 9}{x^2 + 2x - 3}$

8. $\lim_{t \rightarrow 2} \frac{t^2 - 4}{t^3 - 8}$

10. $\lim_{v \rightarrow 4^+} \frac{4 - v}{|4 - v|}$

12. $\lim_{x \rightarrow 3} \frac{\sqrt{x+6} - x}{x^3 - 3x^2}$

14. $\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 - 9}}{2x - 6}$

16. $\lim_{x \rightarrow -\infty} \frac{1 - 2x^2 - x^4}{5 + x - 3x^4}$

17. $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 4x + 1} - x)$

18. $\lim_{x \rightarrow \infty} e^{x-x^2}$

19. $\lim_{x \rightarrow 0^+} \tan^{-1}(1/x)$

20. $\lim_{x \rightarrow 1} \left(\frac{1}{x-1} + \frac{1}{x^2 - 3x + 2} \right)$

- 21–22 Use graphs to discover the asymptotes of the curve. Then prove what you have discovered.

21. $y = \frac{\cos^2 x}{x^2}$

22. $y = \sqrt{x^2 + x + 1} - \sqrt{x^2 - x}$

23. If $2x - 1 \leq f(x) \leq x^2$ for $0 < x < 3$, find $\lim_{x \rightarrow 1} f(x)$.

24. Prove that $\lim_{x \rightarrow 0} x^2 \cos(1/x^2) = 0$.

- 25–28 Prove the statement using the precise definition of a limit.

25. $\lim_{x \rightarrow 2} (14 - 5x) = 4$

26. $\lim_{x \rightarrow 0} \sqrt[3]{x} = 0$

27. $\lim_{x \rightarrow 2} (x^2 - 3x) = -2$

28. $\lim_{x \rightarrow 4^+} \frac{2}{\sqrt{x-4}} = \infty$

29. Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3 - x & \text{if } 0 \leq x < 3 \\ (x-3)^2 & \text{if } x > 3 \end{cases}$$

- (a) Evaluate each limit, if it exists.

(i) $\lim_{x \rightarrow 0^+} f(x)$ (ii) $\lim_{x \rightarrow 0^-} f(x)$ (iii) $\lim_{x \rightarrow 0} f(x)$

(iv) $\lim_{x \rightarrow 3^-} f(x)$ (v) $\lim_{x \rightarrow 3^+} f(x)$ (vi) $\lim_{x \rightarrow 3} f(x)$

- (b) Where is f discontinuous?

- (c) Sketch the graph of f .

30. Let

$$g(x) = \begin{cases} 2x - x^2 & \text{if } 0 \leq x \leq 2 \\ 2 - x & \text{if } 2 < x \leq 3 \\ x - 4 & \text{if } 3 < x < 4 \\ \pi & \text{if } x \geq 4 \end{cases}$$

- (a) For each of the numbers 2, 3, and 4, discover whether g is continuous from the left, continuous from the right, or continuous at the number.

- (b) Sketch the graph of g .

31–32 Show that the function is continuous on its domain. State the domain.

31. $h(x) = xe^{\sin x}$ 32. $g(x) = \frac{\sqrt{x^2 - 9}}{x^2 - 2}$

33–34 Use the Intermediate Value Theorem to show that there is a root of the equation in the given interval.

33. $x^5 - x^3 + 3x - 5 = 0$, $(1, 2)$

34. $\cos \sqrt{x} = e^x - 2$, $(0, 1)$

35. (a) Find the slope of the tangent line to the curve $y = 9 - 2x^2$ at the point $(2, 1)$.
(b) Find an equation of this tangent line.

36. Find equations of the tangent lines to the curve

$$y = \frac{2}{1 - 3x}$$

at the points with x -coordinates 0 and -1 .

37. The displacement (in meters) of an object moving in a straight line is given by $s = 1 + 2t + \frac{1}{4}t^2$, where t is measured in seconds.

- (a) Find the average velocity over each time period.

- (i) $[1, 3]$ (ii) $[1, 2]$
(iii) $[1, 1.5]$ (iv) $[1, 1.1]$

- (b) Find the instantaneous velocity when $t = 1$.

38. According to Boyle's Law, if the temperature of a confined gas is held fixed, then the product of the pressure P and the volume V is a constant. Suppose that, for a certain gas, $PV = 800$, where P is measured in pounds per square inch and V is measured in cubic inches.

- (a) Find the average rate of change of P as V increases from 200 in^3 to 250 in^3 .

- (b) Express V as a function of P and show that the instantaneous rate of change of V with respect to P is inversely proportional to the square of P .

39. (a) Use the definition of a derivative to find $f'(2)$, where $f(x) = x^3 - 2x$.

- (b) Find an equation of the tangent line to the curve $y = x^3 - 2x$ at the point $(2, 4)$.



- (c) Illustrate part (b) by graphing the curve and the tangent line on the same screen.

40. Find a function f and a number a such that

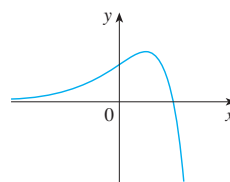
$$\lim_{h \rightarrow 0} \frac{(2 + h)^6 - 64}{h} = f'(a)$$

41. The total cost of repaying a student loan at an interest rate of $r\%$ per year is $C = f(r)$.

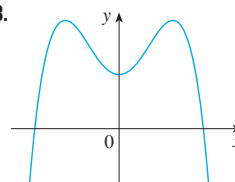
- (a) What is the meaning of the derivative $f'(r)$? What are its units?
(b) What does the statement $f'(10) = 1200$ mean?
(c) Is $f'(r)$ always positive or does it change sign?

42–44 Trace or copy the graph of the function. Then sketch a graph of its derivative directly beneath.

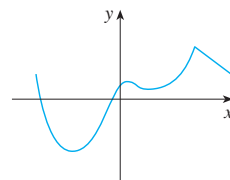
42.



43.



44.



45. (a) If $f(x) = \sqrt{3 - 5x}$, use the definition of a derivative to find $f'(x)$.

- (b) Find the domains of f and f' .



- (c) Graph f and f' on a common screen. Compare the graphs to see whether your answer to part (a) is reasonable.

46. (a) Find the asymptotes of the graph of $f(x) = \frac{4 - x}{3 + x}$ and use them to sketch the graph.

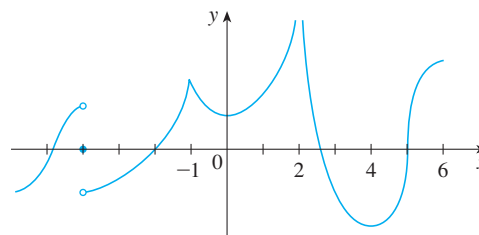
- (b) Use your graph from part (a) to sketch the graph of f' .

- (c) Use the definition of a derivative to find $f'(x)$.

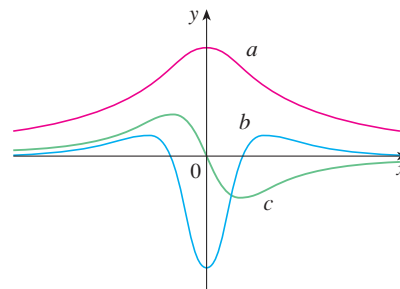


- (d) Use a graphing device to graph f' and compare with your sketch in part (b).

47. The graph of f is shown. State, with reasons, the numbers at which f is not differentiable.



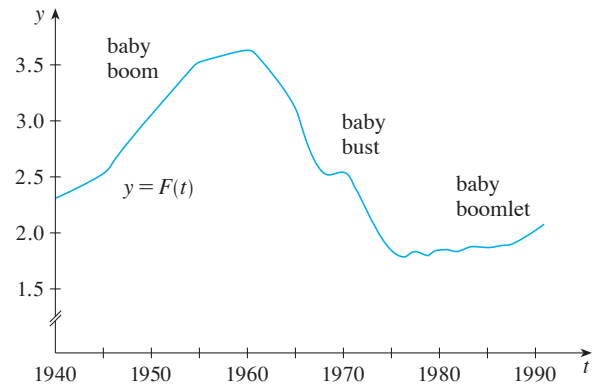
48. The figure shows the graphs of f , f' , and f'' . Identify each curve, and explain your choices.



49. Let $C(t)$ be the total value of US currency (coins and banknotes) in circulation at time t . The table gives values of this function from 1980 to 2000, as of September 30, in billions of dollars. Interpret and estimate the value of $C'(1990)$.

t	1980	1985	1990	1995	2000
$C(t)$	129.9	187.3	271.9	409.3	568.6

50. The *total fertility rate* at time t , denoted by $F(t)$, is an estimate of the average number of children born to each woman (assuming that current birth rates remain constant). The graph of the total fertility rate in the United States shows the fluctuations from 1940 to 1990.
- Estimate the values of $F'(1950)$, $F'(1965)$, and $F'(1987)$.
 - What are the meanings of these derivatives?
 - Can you suggest reasons for the values of these derivatives?



51. Suppose that $|f(x)| \leq g(x)$ for all x , where $\lim_{x \rightarrow a} g(x) = 0$. Find $\lim_{x \rightarrow a} f(x)$.
52. Let $f(x) = \llbracket x \rrbracket + \llbracket -x \rrbracket$.
- For what values of a does $\lim_{x \rightarrow a} f(x)$ exist?
 - At what numbers is f discontinuous?

In our discussion of the principles of problem solving we considered the problem-solving strategy of *introducing something extra* (see page 75). In the following example we show how this principle is sometimes useful when we evaluate limits. The idea is to change the variable—to introduce a new variable that is related to the original variable—in such a way as to make the problem simpler. Later, in Section 5.5, we will make more extensive use of this general idea.

EXAMPLE 1 Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt[3]{1+cx} - 1}{x}$, where c is a constant.

SOLUTION As it stands, this limit looks challenging. In Section 2.3 we evaluated several limits in which both numerator and denominator approached 0. There our strategy was to perform some sort of algebraic manipulation that led to a simplifying cancellation, but here it's not clear what kind of algebra is necessary.

So we introduce a new variable t by the equation

$$t = \sqrt[3]{1+cx}$$

We also need to express x in terms of t , so we solve this equation:

$$t^3 = 1 + cx \quad x = \frac{t^3 - 1}{c} \quad (\text{if } c \neq 0)$$

Notice that $x \rightarrow 0$ is equivalent to $t \rightarrow 1$. This allows us to convert the given limit into one involving the variable t :

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt[3]{1+cx} - 1}{x} &= \lim_{t \rightarrow 1} \frac{t - 1}{(t^3 - 1)/c} \\ &= \lim_{t \rightarrow 1} \frac{c(t - 1)}{t^3 - 1} \end{aligned}$$

The change of variable allowed us to replace a relatively complicated limit by a simpler one of a type that we have seen before. Factoring the denominator as a difference of cubes, we get

$$\begin{aligned} \lim_{t \rightarrow 1} \frac{c(t - 1)}{t^3 - 1} &= \lim_{t \rightarrow 1} \frac{c(t - 1)}{(t - 1)(t^2 + t + 1)} \\ &= \lim_{t \rightarrow 1} \frac{c}{t^2 + t + 1} = \frac{c}{3} \end{aligned}$$

In making the change of variable we had to rule out the case $c = 0$. But if $c = 0$, the function is 0 for all nonzero x and so its limit is 0. Therefore, in all cases, the limit is $c/3$.

The following problems are meant to test and challenge your problem-solving skills. Some of them require a considerable amount of time to think through, so don't be discouraged if you can't solve them right away. If you get stuck, you might find it helpful to refer to the discussion of the principles of problem solving on page 75.

Problems

1. Evaluate $\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1}$.

2. Find numbers a and b such that $\lim_{x \rightarrow 0} \frac{\sqrt{ax+b} - 2}{x} = 1$.

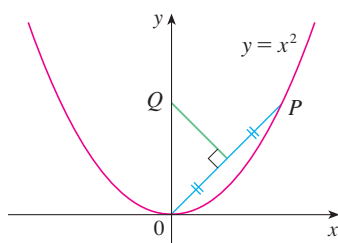


FIGURE FOR PROBLEM 4

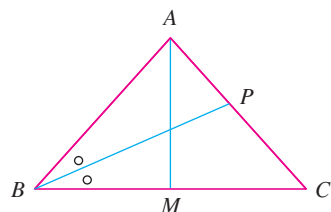


FIGURE FOR PROBLEM 10

3. Evaluate $\lim_{x \rightarrow 0} \frac{|2x - 1| - |2x + 1|}{x}$.

4. The figure shows a point P on the parabola $y = x^2$ and the point Q where the perpendicular bisector of OP intersects the y -axis. As P approaches the origin along the parabola, what happens to Q ? Does it have a limiting position? If so, find it.

5. Evaluate the following limits, if they exist, where $\llbracket x \rrbracket$ denotes the greatest integer function.

(a) $\lim_{x \rightarrow 0} \frac{\llbracket x \rrbracket}{x}$ (b) $\lim_{x \rightarrow 0} x \llbracket 1/x \rrbracket$

6. Sketch the region in the plane defined by each of the following equations.

(a) $\llbracket x \rrbracket^2 + \llbracket y \rrbracket^2 = 1$ (b) $\llbracket x \rrbracket^2 - \llbracket y \rrbracket^2 = 3$ (c) $\llbracket x + y \rrbracket = 1$ (d) $\llbracket x \rrbracket + \llbracket y \rrbracket = 1$

7. Find all values of a such that f is continuous on \mathbb{R} :

$$f(x) = \begin{cases} x + 1 & \text{if } x \leq a \\ x^2 & \text{if } x > a \end{cases}$$

8. A **fixed point** of a function f is a number c in its domain such that $f(c) = c$. (The function doesn't move c ; it stays fixed.)

(a) Sketch the graph of a continuous function with domain $[0, 1]$ whose range also lies in $[0, 1]$. Locate a fixed point of f .

(b) Try to draw the graph of a continuous function with domain $[0, 1]$ and range in $[0, 1]$ that does *not* have a fixed point. What is the obstacle?

(c) Use the Intermediate Value Theorem to prove that any continuous function with domain $[0, 1]$ and range in $[0, 1]$ must have a fixed point.

9. If $\lim_{x \rightarrow a} [f(x) + g(x)] = 2$ and $\lim_{x \rightarrow a} [f(x) - g(x)] = 1$, find $\lim_{x \rightarrow a} [f(x)g(x)]$.

10. (a) The figure shows an isosceles triangle ABC with $\angle B = \angle C$. The bisector of angle B intersects the side AC at the point P . Suppose that the base BC remains fixed but the altitude $|AM|$ of the triangle approaches 0, so A approaches the midpoint M of BC . What happens to P during this process? Does it have a limiting position? If so, find it.
(b) Try to sketch the path traced out by P during this process. Then find an equation of this curve and use this equation to sketch the curve.

11. (a) If we start from 0° latitude and proceed in a westerly direction, we can let $T(x)$ denote the temperature at the point x at any given time. Assuming that T is a continuous function of x , show that at any fixed time there are at least two diametrically opposite points on the equator that have exactly the same temperature.
(b) Does the result in part (a) hold for points lying on any circle on the earth's surface?
(c) Does the result in part (a) hold for barometric pressure and for altitude above sea level?

12. If f is a differentiable function and $g(x) = xf(x)$, use the definition of a derivative to show that $g'(x) = xf'(x) + f(x)$.

13. Suppose f is a function that satisfies the equation

$$f(x + y) = f(x) + f(y) + x^2y + xy^2$$

for all real numbers x and y . Suppose also that

$$\lim_{x \rightarrow 0} \frac{f(x)}{x} = 1$$

(a) Find $f(0)$. (b) Find $f'(0)$. (c) Find $f'(x)$.

14. Suppose f is a function with the property that $|f(x)| \leq x^2$ for all x . Show that $f(0) = 0$. Then show that $f'(0) = 0$.

3

Differentiation Rules

For a roller coaster ride to be smooth, the straight stretches of the track need to be connected to the curved segments so that there are no abrupt changes in direction. In the project on page 184 you will see how to design the first ascent and drop of a new coaster for a smooth ride.



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We have seen how to interpret derivatives as slopes and rates of change. We have seen how to estimate derivatives of functions given by tables of values. We have learned how to graph derivatives of functions that are defined graphically. We have used the definition of a derivative to calculate the derivatives of functions defined by formulas. But it would be tedious if we always had to use the definition, so in this chapter we develop rules for finding derivatives without having to use the definition directly. These differentiation rules enable us to calculate with relative ease the derivatives of polynomials, rational functions, algebraic functions, exponential and logarithmic functions, and trigonometric and inverse trigonometric functions. We then use these rules to solve problems involving rates of change and the approximation of functions.

3.1 Derivatives of Polynomials and Exponential Functions

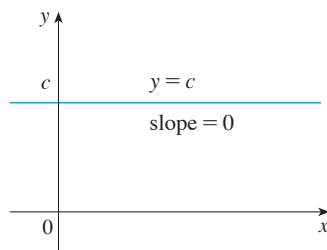


FIGURE 1

The graph of $f(x) = c$ is the line $y = c$, so $f'(x) = 0$.

In this section we learn how to differentiate constant functions, power functions, polynomials, and exponential functions.

Let's start with the simplest of all functions, the constant function $f(x) = c$. The graph of this function is the horizontal line $y = c$, which has slope 0, so we must have $f'(x) = 0$. (See Figure 1.) A formal proof, from the definition of a derivative, is also easy:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} 0 = 0$$

In Leibniz notation, we write this rule as follows.

Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

Power Functions

We next look at the functions $f(x) = x^n$, where n is a positive integer. If $n = 1$, the graph of $f(x) = x$ is the line $y = x$, which has slope 1. (See Figure 2.) So

1

$$\frac{d}{dx}(x) = 1$$

(You can also verify Equation 1 from the definition of a derivative.) We have already investigated the cases $n = 2$ and $n = 3$. In fact, in Section 2.8 (Exercises 19 and 20) we found that

2

$$\frac{d}{dx}(x^2) = 2x \quad \frac{d}{dx}(x^3) = 3x^2$$

For $n = 4$ we find the derivative of $f(x) = x^4$ as follows:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^4 - x^4}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} \\ &= \lim_{h \rightarrow 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\ &= \lim_{h \rightarrow 0} (4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3 \end{aligned}$$

Thus

3

$$\frac{d}{dx}(x^4) = 4x^3$$

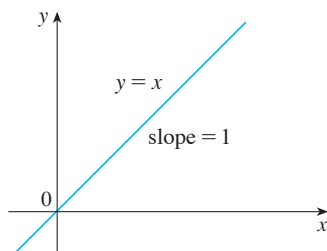


FIGURE 2

The graph of $f(x) = x$ is the line $y = x$, so $f'(x) = 1$.

Comparing the equations in [1], [2], and [3], we see a pattern emerging. It seems to be a reasonable guess that, when n is a positive integer, $(d/dx)(x^n) = nx^{n-1}$. This turns out to be true.

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

FIRST PROOF The formula

$$x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})$$

can be verified simply by multiplying out the right-hand side (or by summing the second factor as a geometric series). If $f(x) = x^n$, we can use Equation 2.7.5 for $f'(a)$ and the equation above to write

$$\begin{aligned} f'(a) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} \\ &= \lim_{x \rightarrow a} (x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) \\ &= a^{n-1} + a^{n-2}a + \cdots + aa^{n-2} + a^{n-1} \\ &= na^{n-1} \end{aligned}$$

SECOND PROOF

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h}$$

The Binomial Theorem is given on Reference Page 1.

In finding the derivative of x^4 we had to expand $(x+h)^4$. Here we need to expand $(x+h)^n$ and we use the Binomial Theorem to do so:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n \right] - x^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\ &= \lim_{h \rightarrow 0} \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1} \right] \\ &= nx^{n-1} \end{aligned}$$

because every term except the first has h as a factor and therefore approaches 0.

We illustrate the Power Rule using various notations in Example 1.

EXAMPLE 1

- (a) If $f(x) = x^6$, then $f'(x) = 6x^5$. (b) If $y = x^{1000}$, then $y' = 1000x^{999}$.
 (c) If $y = t^4$, then $\frac{dy}{dt} = 4t^3$. (d) $\frac{d}{dr}(r^3) = 3r^2$

What about power functions with negative integer exponents? In Exercise 61 we ask you to verify from the definition of a derivative that

$$\frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2}$$

We can rewrite this equation as

$$\frac{d}{dx} (x^{-1}) = (-1)x^{-2}$$

and so the Power Rule is true when $n = -1$. In fact, we will show in the next section [Exercise 62(c)] that it holds for all negative integers.

What if the exponent is a fraction? In Example 3 in Section 2.8 we found that

$$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$$

which can be written as

$$\frac{d}{dx} (x^{1/2}) = \frac{1}{2} x^{-1/2}$$

This shows that the Power Rule is true even when $n = \frac{1}{2}$. In fact, we will show in Section 3.6 that it is true for all real numbers n .

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx} (x^n) = nx^{n-1}$$

Figure 3 shows the function y in Example 2(b) and its derivative y' . Notice that y is not differentiable at 0 (y' is not defined there). Observe that y' is positive when y increases and is negative when y decreases.

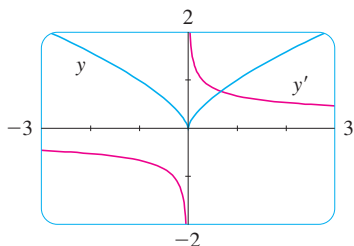


FIGURE 3
 $y = \sqrt[3]{x^2}$

EXAMPLE 2 Differentiate:

(a) $f(x) = \frac{1}{x^2}$

(b) $y = \sqrt[3]{x^2}$

SOLUTION In each case we rewrite the function as a power of x .

(a) Since $f(x) = x^{-2}$, we use the Power Rule with $n = -2$:

$$f'(x) = \frac{d}{dx} (x^{-2}) = -2x^{-2-1} = -2x^{-3} = -\frac{2}{x^3}$$

(b)
$$\frac{dy}{dx} = \frac{d}{dx} (\sqrt[3]{x^2}) = \frac{d}{dx} (x^{2/3}) = \frac{2}{3} x^{(2/3)-1} = \frac{2}{3} x^{-1/3}$$

The Power Rule enables us to find tangent lines without having to resort to the definition of a derivative. It also enables us to find *normal lines*. The **normal line** to a curve C at a point P is the line through P that is perpendicular to the tangent line at P . (In the study of optics, one needs to consider the angle between a light ray and the normal line to a lens.)

V EXAMPLE 3 Find equations of the tangent line and normal line to the curve $y = x\sqrt{x}$ at the point $(1, 1)$. Illustrate by graphing the curve and these lines.

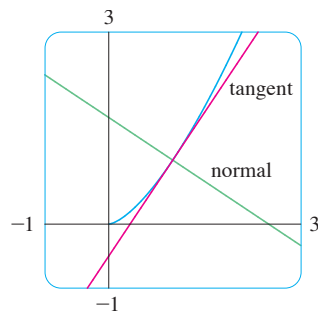


FIGURE 4

$$y = x\sqrt{x}$$

SOLUTION The derivative of $f(x) = x\sqrt{x} = xx^{1/2} = x^{3/2}$ is

$$f'(x) = \frac{3}{2}x^{(3/2)-1} = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x}$$

So the slope of the tangent line at $(1, 1)$ is $f'(1) = \frac{3}{2}$. Therefore an equation of the tangent line is

$$y - 1 = \frac{3}{2}(x - 1) \quad \text{or} \quad y = \frac{3}{2}x - \frac{1}{2}$$

The normal line is perpendicular to the tangent line, so its slope is the negative reciprocal of $\frac{3}{2}$, that is, $-\frac{2}{3}$. Thus an equation of the normal line is

$$y - 1 = -\frac{2}{3}(x - 1) \quad \text{or} \quad y = -\frac{2}{3}x + \frac{5}{3}$$

We graph the curve and its tangent line and normal line in Figure 4.

New Derivatives from Old

When new functions are formed from old functions by addition, subtraction, or multiplication by a constant, their derivatives can be calculated in terms of derivatives of the old functions. In particular, the following formula says that *the derivative of a constant times a function is the constant times the derivative of the function*.

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c \frac{d}{dx}f(x)$$

PROOF Let $g(x) = cf(x)$. Then

$$\begin{aligned} g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} \\ &= \lim_{h \rightarrow 0} c \left[\frac{f(x+h) - f(x)}{h} \right] \\ &= c \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (\text{by Law 3 of limits}) \\ &= cf'(x) \end{aligned}$$

EXAMPLE 4

$$(a) \quad \frac{d}{dx}(3x^4) = 3 \frac{d}{dx}(x^4) = 3(4x^3) = 12x^3$$

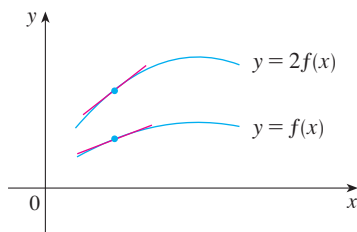
$$(b) \quad \frac{d}{dx}(-x) = \frac{d}{dx}[(-1)x] = (-1) \frac{d}{dx}(x) = -1(1) = -1$$

The next rule tells us that *the derivative of a sum of functions is the sum of the derivatives*.

The Sum Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

GEOMETRIC INTERPRETATION OF THE CONSTANT MULTIPLE RULE



Multiplying by $c = 2$ stretches the graph vertically by a factor of 2. All the rises have been doubled but the runs stay the same. So the slopes are doubled too.

Using prime notation, we can write the Sum Rule as

$$(f + g)' = f' + g'$$

PROOF Let $F(x) = f(x) + g(x)$. Then

$$\begin{aligned}
 F'(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \\
 &= \lim_{h \rightarrow 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right] \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \quad (\text{by Law 1}) \\
 &= f'(x) + g'(x)
 \end{aligned}$$

The Sum Rule can be extended to the sum of any number of functions. For instance, using this theorem twice, we get

$$(f + g + h)' = [(f + g) + h]' = (f + g)' + h' = f' + g' + h'$$

By writing $f - g$ as $f + (-1)g$ and applying the Sum Rule and the Constant Multiple Rule, we get the following formula.

The Difference Rule If f and g are both differentiable, then

$$\frac{d}{dx} [f(x) - g(x)] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

The Constant Multiple Rule, the Sum Rule, and the Difference Rule can be combined with the Power Rule to differentiate any polynomial, as the following examples demonstrate.

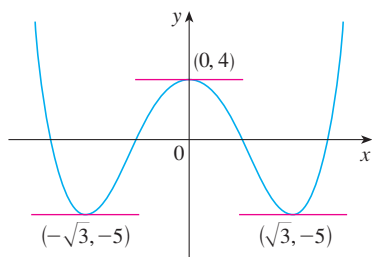
EXAMPLE 5

$$\begin{aligned}
 \frac{d}{dx} (x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5) \\
 &= \frac{d}{dx} (x^8) + 12 \frac{d}{dx} (x^5) - 4 \frac{d}{dx} (x^4) + 10 \frac{d}{dx} (x^3) - 6 \frac{d}{dx} (x) + \frac{d}{dx} (5) \\
 &= 8x^7 + 12(5x^4) - 4(4x^3) + 10(3x^2) - 6(1) + 0 \\
 &= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6
 \end{aligned}$$

V EXAMPLE 6 Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

SOLUTION Horizontal tangents occur where the derivative is zero. We have

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} (x^4) - 6 \frac{d}{dx} (x^2) + \frac{d}{dx} (4) \\
 &= 4x^3 - 12x + 0 = 4x(x^2 - 3)
 \end{aligned}$$


FIGURE 5

The curve $y = x^4 - 6x^2 + 4$ and its horizontal tangents

Thus $dy/dx = 0$ if $x = 0$ or $x^2 - 3 = 0$, that is, $x = \pm\sqrt{3}$. So the given curve has horizontal tangents when $x = 0, \sqrt{3}$, and $-\sqrt{3}$. The corresponding points are $(0, 4)$, $(\sqrt{3}, -5)$, and $(-\sqrt{3}, -5)$. (See Figure 5.)

EXAMPLE 7 The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$, where s is measured in centimeters and t in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

SOLUTION The velocity and acceleration are

$$v(t) = \frac{ds}{dt} = 6t^2 - 10t + 3$$

$$a(t) = \frac{dv}{dt} = 12t - 10$$

The acceleration after 2 s is $a(2) = 14$ cm/s².

Exponential Functions

Let's try to compute the derivative of the exponential function $f(x) = a^x$ using the definition of a derivative:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^x a^h - a^x}{h} = \lim_{h \rightarrow 0} \frac{a^x(a^h - 1)}{h} \end{aligned}$$

The factor a^x doesn't depend on h , so we can take it in front of the limit:

$$f'(x) = a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$$

Notice that the limit is the value of the derivative of f at 0, that is,

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = f'(0)$$

Therefore we have shown that if the exponential function $f(x) = a^x$ is differentiable at 0, then it is differentiable everywhere and

$$f'(x) = f'(0)a^x$$

This equation says that *the rate of change of any exponential function is proportional to the function itself*. (The slope is proportional to the height.)

Numerical evidence for the existence of $f'(0)$ is given in the table at the left for the cases $a = 2$ and $a = 3$. (Values are stated correct to four decimal places.) It appears that the limits exist and

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

$$\text{for } a = 2, \quad f'(0) = \lim_{h \rightarrow 0} \frac{2^h - 1}{h} \approx 0.69$$

$$\text{for } a = 3, \quad f'(0) = \lim_{h \rightarrow 0} \frac{3^h - 1}{h} \approx 1.10$$

In fact, it can be proved that these limits exist and, correct to six decimal places, the values are

$$\left. \frac{d}{dx} (2^x) \right|_{x=0} \approx 0.693147 \quad \left. \frac{d}{dx} (3^x) \right|_{x=0} \approx 1.098612$$

Thus, from Equation 4, we have

$$\boxed{5} \quad \frac{d}{dx} (2^x) \approx (0.69)2^x \quad \frac{d}{dx} (3^x) \approx (1.10)3^x$$

Of all possible choices for the base a in Equation 4, the simplest differentiation formula occurs when $f'(0) = 1$. In view of the estimates of $f'(0)$ for $a = 2$ and $a = 3$, it seems reasonable that there is a number a between 2 and 3 for which $f'(0) = 1$. It is traditional to denote this value by the letter e . (In fact, that is how we introduced e in Section 1.5.) Thus we have the following definition.

Definition of the Number e

$$e \text{ is the number such that } \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

Geometrically, this means that of all the possible exponential functions $y = a^x$, the function $f(x) = e^x$ is the one whose tangent line at $(0, 1)$ has a slope $f'(0)$ that is exactly 1. (See Figures 6 and 7.)

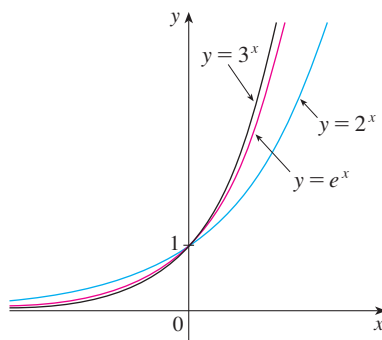


FIGURE 6

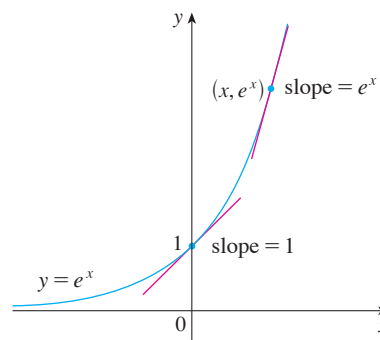


FIGURE 7

If we put $a = e$ and, therefore, $f'(0) = 1$ in Equation 4, it becomes the following important differentiation formula.

Derivative of the Natural Exponential Function

$$\frac{d}{dx} (e^x) = e^x$$

Thus the exponential function $f(x) = e^x$ has the property that it is its own derivative. The geometrical significance of this fact is that the slope of a tangent line to the curve $y = e^x$ is equal to the y -coordinate of the point (see Figure 7).

In Exercise 1 we will see that e lies between 2.7 and 2.8. Later we will be able to show that, correct to five decimal places,

$$e \approx 2.71828$$

TEC Visual 3.1 uses the slope-a-scope to illustrate this formula.

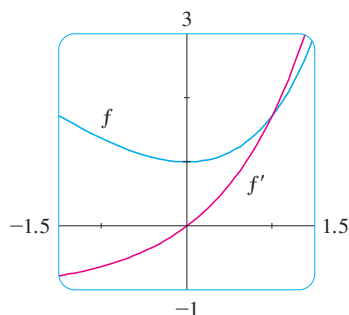


FIGURE 8

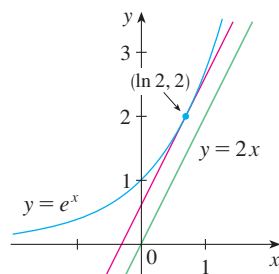


FIGURE 9

V EXAMPLE 8 If $f(x) = e^x - x$, find f' and f'' . Compare the graphs of f and f' .

SOLUTION Using the Difference Rule, we have

$$f'(x) = \frac{d}{dx}(e^x - x) = \frac{d}{dx}(e^x) - \frac{d}{dx}(x) = e^x - 1$$

In Section 2.8 we defined the second derivative as the derivative of f' , so

$$f''(x) = \frac{d}{dx}(e^x - 1) = \frac{d}{dx}(e^x) - \frac{d}{dx}(1) = e^x$$

The function f and its derivative f' are graphed in Figure 8. Notice that f has a horizontal tangent when $x = 0$; this corresponds to the fact that $f'(0) = 0$. Notice also that, for $x > 0$, $f'(x)$ is positive and f is increasing. When $x < 0$, $f'(x)$ is negative and f is decreasing.

EXAMPLE 9 At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

SOLUTION Since $y = e^x$, we have $y' = e^x$. Let the x -coordinate of the point in question be a . Then the slope of the tangent line at that point is e^a . This tangent line will be parallel to the line $y = 2x$ if it has the same slope, that is, 2. Equating slopes, we get

$$e^a = 2 \quad a = \ln 2$$

Therefore the required point is $(a, e^a) = (\ln 2, 2)$. (See Figure 9.)

3.1 Exercises

- (a) How is the number e defined?
(b) Use a calculator to estimate the values of the limits

$$\lim_{h \rightarrow 0} \frac{2.7^h - 1}{h} \quad \text{and} \quad \lim_{h \rightarrow 0} \frac{2.8^h - 1}{h}$$

correct to two decimal places. What can you conclude about the value of e ?

- (a) Sketch, by hand, the graph of the function $f(x) = e^x$, paying particular attention to how the graph crosses the y -axis. What fact allows you to do this?
(b) What types of functions are $f(x) = e^x$ and $g(x) = e^x$? Compare the differentiation formulas for f and g .
(c) Which of the two functions in part (b) grows more rapidly when x is large?

3–32 Differentiate the function.

- $f(x) = 2^{40}$
- $f(t) = 2 - \frac{2}{3}t$
- $f(x) = x^3 - 4x + 6$
- $f(x) = e^5$
- $F(x) = \frac{3}{4}x^8$
- $f(t) = 1.4t^5 - 2.5t^2 + 6.7$

- $g(x) = x^2(1 - 2x)$
- $g(t) = 2t^{-3/4}$
- $A(s) = -\frac{12}{s^5}$
- $R(a) = (3a + 1)^2$
- $S(p) = \sqrt{p} - p$
- $y = 3e^x + \frac{4}{\sqrt[3]{x}}$
- $h(u) = Au^3 + Bu^2 + Cu$
- $y = \frac{x^2 + 4x + 3}{\sqrt{x}}$
- $j(x) = x^{2.4} + e^{2.4}$
- $H(x) = (x + x^{-1})^3$
- $h(x) = (x - 2)(2x + 3)$
- $B(y) = cy^{-6}$
- $y = x^{5/3} - x^{2/3}$
- $h(t) = \sqrt[4]{t} - 4e^t$
- $y = \sqrt{x}(x - 1)$
- $S(R) = 4\pi R^2$
- $y = \frac{\sqrt{x} + x}{x^2}$
- $g(u) = \sqrt{2}u + \sqrt{3}u$
- $k(r) = e^r + r^e$
- $y = ae^v + \frac{b}{v} + \frac{c}{v^2}$



Graphing calculator or computer required

1. Homework Hints available at stewartcalculus.com

$$29. u = \sqrt[3]{t} + 4\sqrt{t^5} \quad 30. v = \left(\sqrt{x} + \frac{1}{\sqrt[3]{x}} \right)^2$$


$$31. z = \frac{A}{y^{10}} + Be^y \quad 32. y = e^{x+1} + 1$$

33–34 Find an equation of the tangent line to the curve at the given point.


$$33. y = \sqrt[4]{x}, \quad (1, 1) \quad 34. y = x^4 + 2x^2 - x, \quad (1, 2)$$

35–36 Find equations of the tangent line and normal line to the curve at the given point.


$$35. y = x^4 + 2e^x, \quad (0, 2) \quad 36. y = x^2 - x^4, \quad (1, 0)$$


 **37–38** Find an equation of the tangent line to the curve at the given point. Illustrate by graphing the curve and the tangent line on the same screen.

$$37. y = 3x^2 - x^3, \quad (1, 2) \quad 38. y = x - \sqrt{x}, \quad (1, 0)$$

 **39–40** Find $f'(x)$. Compare the graphs of f and f' and use them to explain why your answer is reasonable.


$$39. f(x) = x^4 - 2x^3 + x^2 \quad 40. f(x) = x^5 - 2x^3 + x - 1$$

 **41.** (a) Use a graphing calculator or computer to graph the function $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30$ in the viewing rectangle $[-3, 5]$ by $[-10, 50]$.
 (b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of f' . (See Example 1 in Section 2.8.)
 (c) Calculate $f'(x)$ and use this expression, with a graphing device, to graph f' . Compare with your sketch in part (b).

 **42.** (a) Use a graphing calculator or computer to graph the function $g(x) = e^x - 3x^2$ in the viewing rectangle $[-1, 4]$ by $[-8, 8]$.
 (b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of g' . (See Example 1 in Section 2.8.)
 (c) Calculate $g'(x)$ and use this expression, with a graphing device, to graph g' . Compare with your sketch in part (b).

43–44 Find the first and second derivatives of the function.

$$43. f(x) = 10x^{10} + 5x^5 - x \quad 44. G(r) = \sqrt{r} + \sqrt[3]{r}$$

 **45–46** Find the first and second derivatives of the function. Check to see that your answers are reasonable by comparing the graphs of f , f' , and f'' .

$$45. f(x) = 2x - 5x^{3/4} \quad 46. f(x) = e^x - x^3$$

47. The equation of motion of a particle is $s = t^3 - 3t$, where s is in meters and t is in seconds. Find
 (a) the velocity and acceleration as functions of t ,
 (b) the acceleration after 2 s, and
 (c) the acceleration when the velocity is 0.

48. The equation of motion of a particle is $s = t^4 - 2t^3 + t^2 - t$, where s is in meters and t is in seconds.

- (a) Find the velocity and acceleration as functions of t .
 (b) Find the acceleration after 1 s.
 (c) Graph the position, velocity, and acceleration functions on the same screen.



49. Boyle's Law states that when a sample of gas is compressed at a constant pressure, the pressure P of the gas is inversely proportional to the volume V of the gas.

- (a) Suppose that the pressure of a sample of air that occupies 0.106 m^3 at 25°C is 50 kPa. Write V as a function of P .
 (b) Calculate dV/dP when $P = 50$ kPa. What is the meaning of the derivative? What are its units?



50. Car tires need to be inflated properly because overinflation or underinflation can cause premature treadwear. The data in the table show tire life L (in thousands of miles) for a certain type of tire at various pressures P (in lb/in²).

P	26	28	31	35	38	42	45
L	50	66	78	81	74	70	59

- (a) Use a graphing calculator or computer to model tire life with a quadratic function of the pressure.
 (b) Use the model to estimate dL/dP when $P = 30$ and when $P = 40$. What is the meaning of the derivative? What are the units? What is the significance of the signs of the derivatives?

51. Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent is horizontal.

52. For what value of x does the graph of $f(x) = e^x - 2x$ have a horizontal tangent?

53. Show that the curve $y = 2e^x + 3x + 5x^3$ has no tangent line with slope 2.

54. Find an equation of the tangent line to the curve $y = x\sqrt{x}$ that is parallel to the line $y = 1 + 3x$.

55. Find equations of both lines that are tangent to the curve $y = 1 + x^3$ and parallel to the line $12x - y = 1$.



56. At what point on the curve $y = 1 + 2e^x - 3x$ is the tangent line parallel to the line $3x - y = 5$? Illustrate by graphing the curve and both lines.

57. Find an equation of the normal line to the parabola $y = x^2 - 5x + 4$ that is parallel to the line $x - 3y = 5$.

58. Where does the normal line to the parabola $y = x - x^2$ at the point $(1, 0)$ intersect the parabola a second time? Illustrate with a sketch.
59. Draw a diagram to show that there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, -4)$. Find the coordinates of the points where these tangent lines intersect the parabola.
60. (a) Find equations of both lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.
 (b) Show that there is no line through the point $(2, 7)$ that is tangent to the parabola. Then draw a diagram to see why.
61. Use the definition of a derivative to show that if $f(x) = 1/x$, then $f'(x) = -1/x^2$. (This proves the Power Rule for the case $n = -1$.)
62. Find the n th derivative of each function by calculating the first few derivatives and observing the pattern that occurs.
 (a) $f(x) = x^n$ (b) $f(x) = 1/x$
63. Find a second-degree polynomial P such that $P(2) = 5$, $P'(2) = 3$, and $P''(2) = 2$.
64. The equation $y'' + y' - 2y = x^2$ is called a **differential equation** because it involves an unknown function y and its derivatives y' and y'' . Find constants A , B , and C such that the function $y = Ax^2 + Bx + C$ satisfies this equation. (Differential equations will be studied in detail in Chapter 9.)
65. Find a cubic function $y = ax^3 + bx^2 + cx + d$ whose graph has horizontal tangents at the points $(-2, 6)$ and $(2, 0)$.
66. Find a parabola with equation $y = ax^2 + bx + c$ that has slope 4 at $x = 1$, slope -8 at $x = -1$, and passes through the point $(2, 15)$.
67. Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ x + 1 & \text{if } x \geq 1 \end{cases}$$

Is f differentiable at 1? Sketch the graphs of f and f' .

68. At what numbers is the following function g differentiable?

$$g(x) = \begin{cases} 2x & \text{if } x \leq 0 \\ 2x - x^2 & \text{if } 0 < x < 2 \\ 2 - x & \text{if } x \geq 2 \end{cases}$$

Give a formula for g' and sketch the graphs of g and g' .

69. (a) For what values of x is the function $f(x) = |x^2 - 9|$ differentiable? Find a formula for f' .
 (b) Sketch the graphs of f and f' .
70. Where is the function $h(x) = |x - 1| + |x + 2|$ differentiable? Give a formula for h' and sketch the graphs of h and h' .
71. Find the parabola with equation $y = ax^2 + bx$ whose tangent line at $(1, 1)$ has equation $y = 3x - 2$.
72. Suppose the curve $y = x^4 + ax^3 + bx^2 + cx + d$ has a tangent line when $x = 0$ with equation $y = 2x + 1$ and a tangent line when $x = 1$ with equation $y = 2 - 3x$. Find the values of a , b , c , and d .
73. For what values of a and b is the line $2x + y = b$ tangent to the parabola $y = ax^2$ when $x = 2$?
74. Find the value of c such that the line $y = \frac{3}{2}x + 6$ is tangent to the curve $y = c\sqrt{x}$.
75. Let

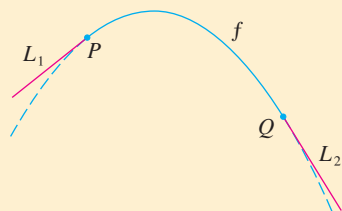
$$f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ mx + b & \text{if } x > 2 \end{cases}$$

Find the values of m and b that make f differentiable everywhere.

76. A tangent line is drawn to the hyperbola $xy = c$ at a point P .
 (a) Show that the midpoint of the line segment cut from this tangent line by the coordinate axes is P .
 (b) Show that the triangle formed by the tangent line and the coordinate axes always has the same area, no matter where P is located on the hyperbola.
77. Evaluate $\lim_{x \rightarrow 1} \frac{x^{1000} - 1}{x - 1}$.
78. Draw a diagram showing two perpendicular lines that intersect on the y -axis and are both tangent to the parabola $y = x^2$. Where do these lines intersect?
79. If $c > \frac{1}{2}$, how many lines through the point $(0, c)$ are normal lines to the parabola $y = x^2$? What if $c \leq \frac{1}{2}$?
80. Sketch the parabolas $y = x^2$ and $y = x^2 - 2x + 2$. Do you think there is a line that is tangent to both curves? If so, find its equation. If not, why not?

APPLIED PROJECT

BUILDING A BETTER ROLLER COASTER



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Suppose you are asked to design the first ascent and drop for a new roller coaster. By studying photographs of your favorite coasters, you decide to make the slope of the ascent 0.8 and the slope of the drop -1.6 . You decide to connect these two straight stretches $y = L_1(x)$ and $y = L_2(x)$ with part of a parabola $y = f(x) = ax^2 + bx + c$, where x and $f(x)$ are measured in feet. For the track to be smooth there can't be abrupt changes in direction, so you want the linear segments L_1 and L_2 to be tangent to the parabola at the transition points P and Q . (See the figure.) To simplify the equations, you decide to place the origin at P .

- Suppose the horizontal distance between P and Q is 100 ft. Write equations in a , b , and c that will ensure that the track is smooth at the transition points.
 - Solve the equations in part (a) for a , b , and c to find a formula for $f(x)$.
 - Plot L_1 , f , and L_2 to verify graphically that the transitions are smooth.
 - Find the difference in elevation between P and Q .
- The solution in Problem 1 might *look* smooth, but it might not *feel* smooth because the piecewise defined function [consisting of $L_1(x)$ for $x < 0$, $f(x)$ for $0 \leq x \leq 100$, and $L_2(x)$ for $x > 100$] doesn't have a continuous second derivative. So you decide to improve the design by using a quadratic function $q(x) = ax^2 + bx + c$ only on the interval $10 \leq x \leq 90$ and connecting it to the linear functions by means of two cubic functions:

$$g(x) = kx^3 + lx^2 + mx + n \quad 0 \leq x < 10$$

$$h(x) = px^3 + qx^2 + rx + s \quad 90 < x \leq 100$$

- Write a system of equations in 11 unknowns that ensure that the functions and their first two derivatives agree at the transition points.
- Solve the equations in part (a) with a computer algebra system to find formulas for $q(x)$, $g(x)$, and $h(x)$.
- Plot L_1 , g , q , h , and L_2 , and compare with the plot in Problem 1(c).



Graphing calculator or computer required



Computer algebra system required

3.2 The Product and Quotient Rules

The formulas of this section enable us to differentiate new functions formed from old functions by multiplication or division.

The Product Rule



By analogy with the Sum and Difference Rules, one might be tempted to guess, as Leibniz did three centuries ago, that the derivative of a product is the product of the derivatives. We can see, however, that this guess is wrong by looking at a particular example. Let $f(x) = x$ and $g(x) = x^2$. Then the Power Rule gives $f'(x) = 1$ and $g'(x) = 2x$. But $(fg)(x) = x^3$, so $(fg)'(x) = 3x^2$. Thus $(fg)' \neq f'g'$. The correct formula was discovered by Leibniz (soon after his false start) and is called the Product Rule.

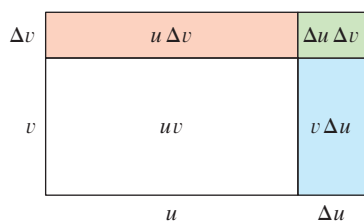


FIGURE 1
The geometry of the Product Rule

Before stating the Product Rule, let's see how we might discover it. We start by assuming that $u = f(x)$ and $v = g(x)$ are both positive differentiable functions. Then we can interpret the product uv as an area of a rectangle (see Figure 1). If x changes by an amount Δx , then the corresponding changes in u and v are

$$\Delta u = f(x + \Delta x) - f(x) \quad \Delta v = g(x + \Delta x) - g(x)$$

and the new value of the product, $(u + \Delta u)(v + \Delta v)$, can be interpreted as the area of the large rectangle in Figure 1 (provided that Δu and Δv happen to be positive).

The change in the area of the rectangle is

$$\begin{aligned} \Delta(uv) &= (u + \Delta u)(v + \Delta v) - uv = u \Delta v + v \Delta u + \Delta u \Delta v \\ &= \text{the sum of the three shaded areas} \end{aligned}$$

If we divide by Δx , we get

$$\frac{\Delta(uv)}{\Delta x} = u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x}$$

Recall that in Leibniz notation the definition of a derivative can be written as

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

If we now let $\Delta x \rightarrow 0$, we get the derivative of uv :

$$\begin{aligned} \frac{d}{dx}(uv) &= \lim_{\Delta x \rightarrow 0} \frac{\Delta(uv)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \left(u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x} \right) \\ &= u \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} + v \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} + \left(\lim_{\Delta x \rightarrow 0} \Delta u \right) \left(\lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} \right) \\ &= u \frac{dv}{dx} + v \frac{du}{dx} + 0 \cdot \frac{dv}{dx} \end{aligned}$$

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

(Notice that $\Delta u \rightarrow 0$ as $\Delta x \rightarrow 0$ since f is differentiable and therefore continuous.)

Although we started by assuming (for the geometric interpretation) that all the quantities are positive, we notice that Equation 1 is always true. (The algebra is valid whether u , v , Δu , and Δv are positive or negative.) So we have proved Equation 2, known as the Product Rule, for all differentiable functions u and v .

In prime notation:

$$(fg)' = fg' + gf'$$

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x) \frac{d}{dx}[g(x)] + g(x) \frac{d}{dx}[f(x)]$$

In words, the Product Rule says that *the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.*

EXAMPLE 1

- (a) If $f(x) = xe^x$, find $f'(x)$.
 (b) Find the n th derivative, $f^{(n)}(x)$.

SOLUTION

- (a) By the Product Rule, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}(xe^x) \\ &= x \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x) \\ &= xe^x + e^x \cdot 1 = (x + 1)e^x \end{aligned}$$

- (b) Using the Product Rule a second time, we get

$$\begin{aligned} f''(x) &= \frac{d}{dx}[(x + 1)e^x] \\ &= (x + 1) \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x + 1) \\ &= (x + 1)e^x + e^x \cdot 1 = (x + 2)e^x \end{aligned}$$

Further applications of the Product Rule give

$$f'''(x) = (x + 3)e^x \quad f^{(4)}(x) = (x + 4)e^x$$

In fact, each successive differentiation adds another term e^x , so

$$f^{(n)}(x) = (x + n)e^x$$

Figure 2 shows the graphs of the function f of Example 1 and its derivative f' . Notice that $f'(x)$ is positive when f is increasing and negative when f is decreasing.

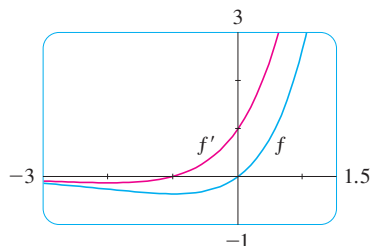


FIGURE 2

In Example 2, a and b are constants. It is customary in mathematics to use letters near the beginning of the alphabet to represent constants and letters near the end of the alphabet to represent variables.

EXAMPLE 2 Differentiate the function $f(t) = \sqrt{t}(a + bt)$.**SOLUTION 1** Using the Product Rule, we have

$$\begin{aligned} f'(t) &= \sqrt{t} \frac{d}{dt}(a + bt) + (a + bt) \frac{d}{dt}(\sqrt{t}) \\ &= \sqrt{t} \cdot b + (a + bt) \cdot \frac{1}{2}t^{-1/2} \\ &= b\sqrt{t} + \frac{a + bt}{2\sqrt{t}} = \frac{a + 3bt}{2\sqrt{t}} \end{aligned}$$

SOLUTION 2 If we first use the laws of exponents to rewrite $f(t)$, then we can proceed directly without using the Product Rule.

$$\begin{aligned} f(t) &= a\sqrt{t} + bt\sqrt{t} = at^{1/2} + bt^{3/2} \\ f'(t) &= \frac{1}{2}at^{-1/2} + \frac{3}{2}bt^{1/2} \end{aligned}$$

which is equivalent to the answer given in Solution 1.

Example 2 shows that it is sometimes easier to simplify a product of functions before differentiating than to use the Product Rule. In Example 1, however, the Product Rule is the only possible method.

EXAMPLE 3 If $f(x) = \sqrt{x} g(x)$, where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

SOLUTION Applying the Product Rule, we get

$$\begin{aligned} f'(x) &= \frac{d}{dx} [\sqrt{x} g(x)] = \sqrt{x} \frac{d}{dx} [g(x)] + g(x) \frac{d}{dx} [\sqrt{x}] \\ &= \sqrt{x} g'(x) + g(x) \cdot \frac{1}{2} x^{-1/2} \\ &= \sqrt{x} g'(x) + \frac{g(x)}{2\sqrt{x}} \end{aligned}$$

So
$$f'(4) = \sqrt{4} g'(4) + \frac{g(4)}{2\sqrt{4}} = 2 \cdot 3 + \frac{2}{2 \cdot 2} = 6.5$$

The Quotient Rule

We find a rule for differentiating the quotient of two differentiable functions $u = f(x)$ and $v = g(x)$ in much the same way that we found the Product Rule. If x , u , and v change by amounts Δx , Δu , and Δv , then the corresponding change in the quotient u/v is

$$\begin{aligned} \Delta\left(\frac{u}{v}\right) &= \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{(u + \Delta u)v - u(v + \Delta v)}{v(v + \Delta v)} \\ &= \frac{v\Delta u - u\Delta v}{v(v + \Delta v)} \end{aligned}$$

so

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{\Delta x \rightarrow 0} \frac{\Delta(u/v)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{v \frac{\Delta u}{\Delta x} - u \frac{\Delta v}{\Delta x}}{v(v + \Delta v)}$$

As $\Delta x \rightarrow 0$, $\Delta v \rightarrow 0$ also, because $v = g(x)$ is differentiable and therefore continuous. Thus, using the Limit Laws, we get

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} - u \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x}}{v \lim_{\Delta x \rightarrow 0} (v + \Delta v)} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

In prime notation:

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} [f(x)] - f(x) \frac{d}{dx} [g(x)]}{[g(x)]^2}$$

In words, the Quotient Rule says that the *derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

The Quotient Rule and the other differentiation formulas enable us to compute the derivative of any rational function, as the next example illustrates.

We can use a graphing device to check that the answer to Example 4 is plausible. Figure 3 shows the graphs of the function of Example 4 and its derivative. Notice that when y grows rapidly (near -2), y' is large. And when y grows slowly, y' is near 0.

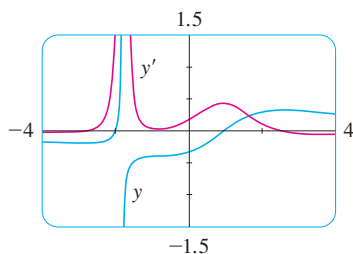


FIGURE 3

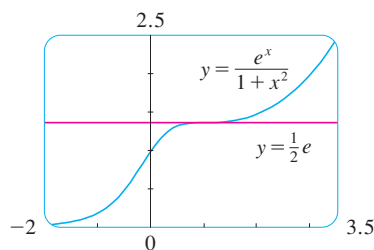


FIGURE 4

V EXAMPLE 4 Let $y = \frac{x^2 + x - 2}{x^3 + 6}$. Then

$$\begin{aligned} y' &= \frac{(x^3 + 6) \frac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2) \frac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2} \\ &= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2} \\ &= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2} \\ &= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2} \end{aligned}$$

V EXAMPLE 5 Find an equation of the tangent line to the curve $y = e^x/(1 + x^2)$ at the point $(1, \frac{1}{2}e)$.

SOLUTION According to the Quotient Rule, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{(1 + x^2) \frac{d}{dx}(e^x) - e^x \frac{d}{dx}(1 + x^2)}{(1 + x^2)^2} \\ &= \frac{(1 + x^2)e^x - e^x(2x)}{(1 + x^2)^2} \\ &= \frac{e^x(1 - x)}{(1 + x^2)^2} \end{aligned}$$

So the slope of the tangent line at $(1, \frac{1}{2}e)$ is

$$\left. \frac{dy}{dx} \right|_{x=1} = 0$$

This means that the tangent line at $(1, \frac{1}{2}e)$ is horizontal and its equation is $y = \frac{1}{2}e$. [See Figure 4. Notice that the function is increasing and crosses its tangent line at $(1, \frac{1}{2}e)$.]

NOTE Don't use the Quotient Rule *every* time you see a quotient. Sometimes it's easier to rewrite a quotient first to put it in a form that is simpler for the purpose of differentiation. For instance, although it is possible to differentiate the function

$$F(x) = \frac{3x^2 + 2\sqrt{x}}{x}$$

using the Quotient Rule, it is much easier to perform the division first and write the function as

$$F(x) = 3x + 2x^{-1/2}$$

before differentiating.

We summarize the differentiation formulas we have learned so far as follows.

Table of Differentiation Formulas

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$(cf)' = cf'$$

$$(f + g)' = f' + g'$$

$$(f - g)' = f' - g'$$

$$(fg)' = fg' + gf'$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

3.2 Exercises

1. Find the derivative of $f(x) = (1 + 2x^2)(x - x^2)$ in two ways: by using the Product Rule and by performing the multiplication first. Do your answers agree?

2. Find the derivative of the function

$$F(x) = \frac{x^4 - 5x^3 + \sqrt{x}}{x^2}$$

in two ways: by using the Quotient Rule and by simplifying first. Show that your answers are equivalent. Which method do you prefer?

- 3–26 Differentiate.

3. $f(x) = (x^3 + 2x)e^x$

4. $g(x) = \sqrt{x} e^x$

5. $y = \frac{x}{e^x}$

6. $y = \frac{e^x}{1 - e^x}$

7. $g(x) = \frac{1 + 2x}{3 - 4x}$

8. $G(x) = \frac{x^2 - 2}{2x + 1}$

9. $H(u) = (u - \sqrt{u})(u + \sqrt{u})$

10. $J(v) = (v^3 - 2v)(v^{-4} + v^{-2})$

11. $F(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3)$

12. $f(z) = (1 - e^z)(z + e^z)$

13. $y = \frac{x^3}{1 - x^2}$

14. $y = \frac{x + 1}{x^3 + x - 2}$

15. $y = \frac{t^2 + 2}{t^4 - 3t^2 + 1}$

16. $y = \frac{t}{(t - 1)^2}$

17. $y = e^p(p + p\sqrt{p})$

18. $y = \frac{1}{s + ke^s}$

19. $y = \frac{v^3 - 2v\sqrt{v}}{v}$

20. $z = w^{3/2}(w + ce^w)$

21. $f(t) = \frac{2t}{2 + \sqrt{t}}$

22. $g(t) = \frac{t - \sqrt{t}}{t^{1/3}}$

23. $f(x) = \frac{A}{B + Ce^x}$

24. $f(x) = \frac{1 - xe^x}{x + e^x}$

25. $f(x) = \frac{x}{x + \frac{c}{x}}$

26. $f(x) = \frac{ax + b}{cx + d}$

- 27–30 Find $f'(x)$ and $f''(x)$.

27. $f(x) = x^4 e^x$

28. $f(x) = x^{5/2} e^x$

29. $f(x) = \frac{x^2}{1 + 2x}$

30. $f(x) = \frac{x}{x^2 - 1}$

- 31–32 Find an equation of the tangent line to the given curve at the specified point.

31. $y = \frac{x^2 - 1}{x^2 + x + 1}, \quad (1, 0)$

32. $y = \frac{e^x}{x}, \quad (1, e)$

- 33–34 Find equations of the tangent line and normal line to the given curve at the specified point.

33. $y = 2xe^x, \quad (0, 0)$

34. $y = \frac{2x}{x^2 + 1}, \quad (1, 1)$

35. (a) The curve $y = 1/(1 + x^2)$ is called a **witch of Maria Agnesi**. Find an equation of the tangent line to this curve at the point $(-1, \frac{1}{2})$.



- (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

36. (a) The curve $y = x/(1 + x^2)$ is called a **serpentine**. Find an equation of the tangent line to this curve at the point $(3, 0.3)$.



- (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

37. (a) If $f(x) = (x^3 - x)e^x$, find $f'(x)$.



- (b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f' .

38. (a) If $f(x) = e^x/(2x^2 + x + 1)$, find $f'(x)$.



- (b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f' .



Graphing calculator or computer required

1. Homework Hints available at stewartcalculus.com

39. (a) If $f(x) = (x^2 - 1)/(x^2 + 1)$, find $f'(x)$ and $f''(x)$.
 (b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f , f' , and f'' .

40. (a) If $f(x) = (x^2 - 1)e^x$, find $f'(x)$ and $f''(x)$.
 (b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f , f' , and f'' .

41. If $f(x) = x^2/(1 + x)$, find $f''(1)$.

42. If $g(x) = x/e^x$, find $g^{(n)}(x)$.

43. Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$. Find the following values.

(a) $(fg)'(5)$ (b) $(f/g)'(5)$ (c) $(g/f)'(5)$

44. Suppose that $f(2) = -3$, $g(2) = 4$, $f'(2) = -2$, and $g'(2) = 7$. Find $h'(2)$.

(a) $h(x) = 5f(x) - 4g(x)$ (b) $h(x) = f(x)g(x)$

(c) $h(x) = \frac{f(x)}{g(x)}$ (d) $h(x) = \frac{g(x)}{1 + f(x)}$

45. If $f(x) = e^x g(x)$, where $g(0) = 2$ and $g'(0) = 5$, find $f'(0)$.

46. If $h(2) = 4$ and $h'(2) = -3$, find

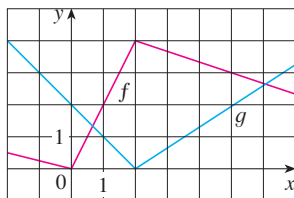
$$\left. \frac{d}{dx} \left(\frac{h(x)}{x} \right) \right|_{x=2}$$

47. If $g(x) = xf(x)$, where $f(3) = 4$ and $f'(3) = -2$, find an equation of the tangent line to the graph of g at the point where $x = 3$.

48. If $f(2) = 10$ and $f'(x) = x^2 f(x)$ for all x , find $f''(2)$.

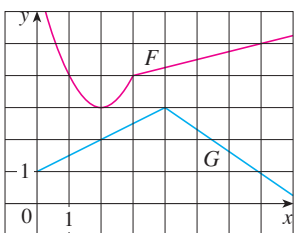
49. If f and g are the functions whose graphs are shown, let $u(x) = f(x)g(x)$ and $v(x) = f(x)/g(x)$.

(a) Find $u'(1)$. (b) Find $v'(5)$.



50. Let $P(x) = F(x)G(x)$ and $Q(x) = F(x)/G(x)$, where F and G are the functions whose graphs are shown.

(a) Find $P'(2)$. (b) Find $Q'(7)$.



51. If g is a differentiable function, find an expression for the derivative of each of the following functions.

(a) $y = xg(x)$ (b) $y = \frac{x}{g(x)}$ (c) $y = \frac{g(x)}{x}$

52. If f is a differentiable function, find an expression for the derivative of each of the following functions.

(a) $y = x^2 f(x)$ (b) $y = \frac{f(x)}{x^2}$

(c) $y = \frac{x^2}{f(x)}$ (d) $y = \frac{1 + xf(x)}{\sqrt{x}}$

53. How many tangent lines to the curve $y = x/(x + 1)$ pass through the point $(1, 2)$? At which points do these tangent lines touch the curve?

54. Find equations of the tangent lines to the curve

$$y = \frac{x - 1}{x + 1}$$

that are parallel to the line $x - 2y = 2$.

55. Find $R'(0)$, where

$$R(x) = \frac{x - 3x^3 + 5x^5}{1 + 3x^3 + 6x^6 + 9x^9}$$

Hint: Instead of finding $R'(x)$ first, let $f(x)$ be the numerator and $g(x)$ the denominator of $R(x)$ and compute $R'(0)$ from $f(0)$, $f'(0)$, $g(0)$, and $g'(0)$.

56. Use the method of Exercise 55 to compute $Q'(0)$, where

$$Q(x) = \frac{1 + x + x^2 + xe^x}{1 - x + x^2 - xe^x}$$

57. In this exercise we estimate the rate at which the total personal income is rising in the Richmond-Petersburg, Virginia, metropolitan area. In 1999, the population of this area was 961,400, and the population was increasing at roughly 9200 people per year. The average annual income was \$30,593 per capita, and this average was increasing at about \$1400 per year (a little above the national average of about \$1225 yearly). Use the Product Rule and these figures to estimate the rate at which total personal income was rising in the Richmond-Petersburg area in 1999. Explain the meaning of each term in the Product Rule.

58. A manufacturer produces bolts of a fabric with a fixed width. The quantity q of this fabric (measured in yards) that is sold is a function of the selling price p (in dollars per yard), so we can write $q = f(p)$. Then the total revenue earned with selling price p is $R(p) = pf(p)$.

(a) What does it mean to say that $f(20) = 10,000$ and $f'(20) = -350$?

(b) Assuming the values in part (a), find $R'(20)$ and interpret your answer.

59. (a) Use the Product Rule twice to prove that if f , g , and h are differentiable, then $(fgh)' = f'gh + fg'h + fgh'$.
 (b) Taking $f = g = h$ in part (a), show that

$$\frac{d}{dx} [f(x)]^3 = 3[f(x)]^2 f'(x)$$

- (c) Use part (b) to differentiate $y = e^{3x}$.
 60. (a) If $F(x) = f(x)g(x)$, where f and g have derivatives of all orders, show that $F'' = f''g + 2f'g' + fg''$.
 (b) Find similar formulas for F''' and $F^{(4)}$.
 (c) Guess a formula for $F^{(n)}$.
 61. Find expressions for the first five derivatives of $f(x) = x^2 e^x$. Do you see a pattern in these expressions? Guess a formula for $f^{(n)}(x)$ and prove it using mathematical induction.

62. (a) If g is differentiable, the **Reciprocal Rule** says that

$$\frac{d}{dx} \left[\frac{1}{g(x)} \right] = -\frac{g'(x)}{[g(x)]^2}$$

- Use the Quotient Rule to prove the Reciprocal Rule.
 (b) Use the Reciprocal Rule to differentiate the function in Exercise 18.
 (c) Use the Reciprocal Rule to verify that the Power Rule is valid for negative integers, that is,

$$\frac{d}{dx} (x^{-n}) = -nx^{-n-1}$$

for all positive integers n .

3.3 Derivatives of Trigonometric Functions

A review of the trigonometric functions is given in Appendix D.

Before starting this section, you might need to review the trigonometric functions. In particular, it is important to remember that when we talk about the function f defined for all real numbers x by

$$f(x) = \sin x$$

it is understood that $\sin x$ means the sine of the angle whose *radian* measure is x . A similar convention holds for the other trigonometric functions \cos , \tan , \csc , \sec , and \cot . Recall from Section 2.5 that all of the trigonometric functions are continuous at every number in their domains.

If we sketch the graph of the function $f(x) = \sin x$ and use the interpretation of $f'(x)$ as the slope of the tangent to the sine curve in order to sketch the graph of f' (see Exercise 16 in Section 2.8), then it looks as if the graph of f' may be the same as the cosine curve (see Figure 1).

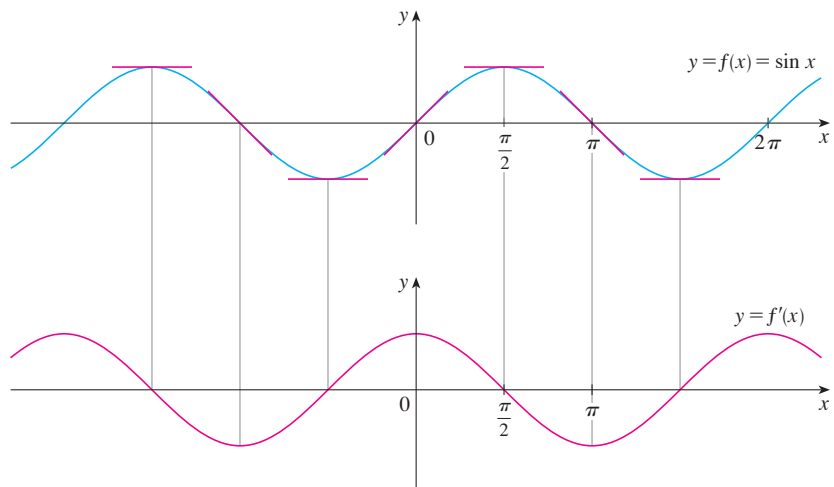


FIGURE 1

TEC Visual 3.3 shows an animation of Figure 1.

Let's try to confirm our guess that if $f(x) = \sin x$, then $f'(x) = \cos x$. From the definition of a derivative, we have

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
 &= \lim_{h \rightarrow 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right] \\
 &= \lim_{h \rightarrow 0} \sin x \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \lim_{h \rightarrow 0} \cos x \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h}
 \end{aligned}$$

We have used the addition formula for sine. See Appendix D.

1

Two of these four limits are easy to evaluate. Since we regard x as a constant when computing a limit as $h \rightarrow 0$, we have

$$\lim_{h \rightarrow 0} \sin x = \sin x \quad \text{and} \quad \lim_{h \rightarrow 0} \cos x = \cos x$$

The limit of $(\sin h)/h$ is not so obvious. In Example 3 in Section 2.2 we made the guess, on the basis of numerical and graphical evidence, that

2

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

We now use a geometric argument to prove Equation 2. Assume first that θ lies between 0 and $\pi/2$. Figure 2(a) shows a sector of a circle with center O , central angle θ , and radius 1. BC is drawn perpendicular to OA . By the definition of radian measure, we have arc $AB = \theta$. Also $|BC| = |OB| \sin \theta = \sin \theta$. From the diagram we see that

$$|BC| < |AB| < \text{arc } AB$$

Therefore $\sin \theta < \theta$ so $\frac{\sin \theta}{\theta} < 1$

Let the tangent lines at A and B intersect at E . You can see from Figure 2(b) that the circumference of a circle is smaller than the length of a circumscribed polygon, and so arc $AB < |AE| + |EB|$. Thus

$$\begin{aligned}
 \theta = \text{arc } AB &< |AE| + |EB| \\
 &< |AE| + |ED| \\
 &= |AD| = |OA| \tan \theta \\
 &= \tan \theta
 \end{aligned}$$

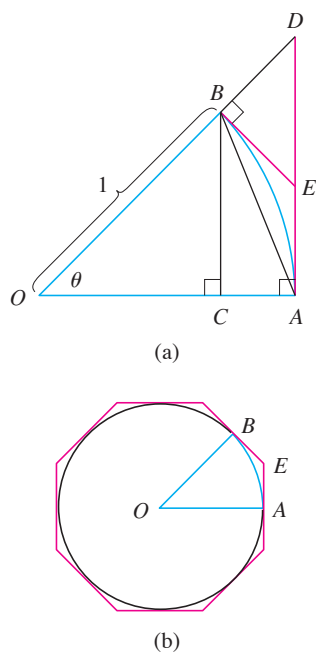


FIGURE 2

(In Appendix F the inequality $\theta \leq \tan \theta$ is proved directly from the definition of the length

of an arc without resorting to geometric intuition as we did here.) Therefore we have

$$\theta < \frac{\sin \theta}{\cos \theta}$$

so
$$\cos \theta < \frac{\sin \theta}{\theta} < 1$$

We know that $\lim_{\theta \rightarrow 0} 1 = 1$ and $\lim_{\theta \rightarrow 0} \cos \theta = 1$, so by the Squeeze Theorem, we have

$$\lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1$$

But the function $(\sin \theta)/\theta$ is an even function, so its right and left limits must be equal. Hence, we have

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

so we have proved Equation 2.

We can deduce the value of the remaining limit in [1] as follows:

We multiply numerator and denominator by $\cos \theta + 1$ in order to put the function in a form in which we can use the limits we know.

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} &= \lim_{\theta \rightarrow 0} \left(\frac{\cos \theta - 1}{\theta} \cdot \frac{\cos \theta + 1}{\cos \theta + 1} \right) = \lim_{\theta \rightarrow 0} \frac{\cos^2 \theta - 1}{\theta (\cos \theta + 1)} \\ &= \lim_{\theta \rightarrow 0} \frac{-\sin^2 \theta}{\theta (\cos \theta + 1)} = -\lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{\cos \theta + 1} \right) \\ &= -\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\cos \theta + 1} \\ &= -1 \cdot \left(\frac{0}{1 + 1} \right) = 0 \quad (\text{by Equation 2}) \end{aligned}$$

3

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0$$

If we now put the limits [2] and [3] in [1], we get

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \sin x \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \lim_{h \rightarrow 0} \cos x \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= (\sin x) \cdot 0 + (\cos x) \cdot 1 = \cos x \end{aligned}$$

So we have proved the formula for the derivative of the sine function:

4

$$\frac{d}{dx} (\sin x) = \cos x$$

Figure 3 shows the graphs of the function of Example 1 and its derivative. Notice that $y' = 0$ whenever y has a horizontal tangent.

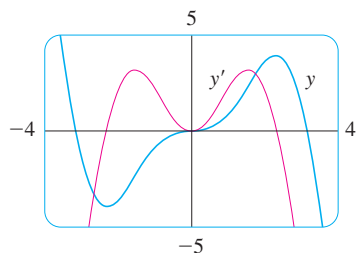


FIGURE 3

V EXAMPLE 1 Differentiate $y = x^2 \sin x$.

SOLUTION Using the Product Rule and Formula 4, we have

$$\begin{aligned}\frac{dy}{dx} &= x^2 \frac{d}{dx} (\sin x) + \sin x \frac{d}{dx} (x^2) \\ &= x^2 \cos x + 2x \sin x\end{aligned}$$

Using the same methods as in the proof of Formula 4, one can prove (see Exercise 20) that

5

$$\frac{d}{dx} (\cos x) = -\sin x$$

The tangent function can also be differentiated by using the definition of a derivative, but it is easier to use the Quotient Rule together with Formulas 4 and 5:

$$\begin{aligned}\frac{d}{dx} (\tan x) &= \frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) \\ &= \frac{\cos x \frac{d}{dx} (\sin x) - \sin x \frac{d}{dx} (\cos x)}{\cos^2 x} \\ &= \frac{\cos x \cdot \cos x - \sin x (-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

6

$$\frac{d}{dx} (\tan x) = \sec^2 x$$

The derivatives of the remaining trigonometric functions, \csc , \sec , and \cot , can also be found easily using the Quotient Rule (see Exercises 17–19). We collect all the differentiation formulas for trigonometric functions in the following table. Remember that they are valid only when x is measured in radians.

Derivatives of Trigonometric Functions

$$\frac{d}{dx} (\sin x) = \cos x$$

$$\frac{d}{dx} (\csc x) = -\csc x \cot x$$

$$\frac{d}{dx} (\cos x) = -\sin x$$

$$\frac{d}{dx} (\sec x) = \sec x \tan x$$

$$\frac{d}{dx} (\tan x) = \sec^2 x$$

$$\frac{d}{dx} (\cot x) = -\csc^2 x$$

When you memorize this table, it is helpful to notice that the minus signs go with the derivatives of the “cofunctions,” that is, cosine, cosecant, and cotangent.

EXAMPLE 2 Differentiate $f(x) = \frac{\sec x}{1 + \tan x}$. For what values of x does the graph of f have a horizontal tangent?

SOLUTION The Quotient Rule gives

$$\begin{aligned} f'(x) &= \frac{(1 + \tan x) \frac{d}{dx}(\sec x) - \sec x \frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2} \\ &= \frac{(1 + \tan x) \sec x \tan x - \sec x \cdot \sec^2 x}{(1 + \tan x)^2} \\ &= \frac{\sec x (\tan x + \tan^2 x - \sec^2 x)}{(1 + \tan x)^2} \\ &= \frac{\sec x (\tan x - 1)}{(1 + \tan x)^2} \end{aligned}$$

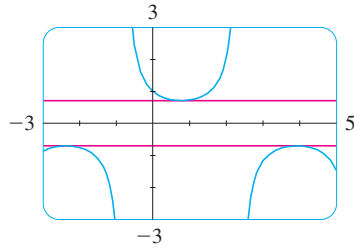


FIGURE 4
The horizontal tangents in Example 2

In simplifying the answer we have used the identity $\tan^2 x + 1 = \sec^2 x$.

Since $\sec x$ is never 0, we see that $f'(x) = 0$ when $\tan x = 1$, and this occurs when $x = n\pi + \pi/4$, where n is an integer (see Figure 4).

Trigonometric functions are often used in modeling real-world phenomena. In particular, vibrations, waves, elastic motions, and other quantities that vary in a periodic manner can be described using trigonometric functions. In the following example we discuss an instance of simple harmonic motion.

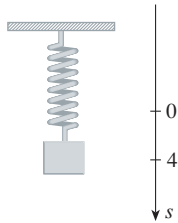


FIGURE 5

EXAMPLE 3 An object at the end of a vertical spring is stretched 4 cm beyond its rest position and released at time $t = 0$. (See Figure 5 and note that the downward direction is positive.) Its position at time t is

$$s = f(t) = 4 \cos t$$

Find the velocity and acceleration at time t and use them to analyze the motion of the object.

SOLUTION The velocity and acceleration are

$$\begin{aligned} v &= \frac{ds}{dt} = \frac{d}{dt}(4 \cos t) = 4 \frac{d}{dt}(\cos t) = -4 \sin t \\ a &= \frac{dv}{dt} = \frac{d}{dt}(-4 \sin t) = -4 \frac{d}{dt}(\sin t) = -4 \cos t \end{aligned}$$

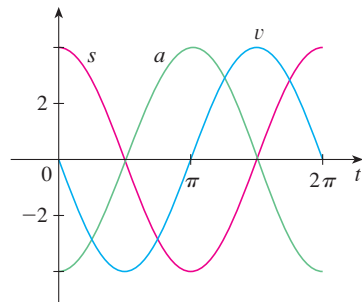


FIGURE 6

The object oscillates from the lowest point ($s = 4$ cm) to the highest point ($s = -4$ cm). The period of the oscillation is 2π , the period of $\cos t$.

The speed is $|v| = 4|\sin t|$, which is greatest when $|\sin t| = 1$, that is, when $\cos t = 0$. So the object moves fastest as it passes through its equilibrium position ($s = 0$). Its speed is 0 when $\sin t = 0$, that is, at the high and low points.

The acceleration $a = -4 \cos t = 0$ when $s = 0$. It has greatest magnitude at the high and low points. See the graphs in Figure 6.

EXAMPLE 4 Find the 27th derivative of $\cos x$.

SOLUTION The first few derivatives of $f(x) = \cos x$ are as follows:

PS Look for a pattern.

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

$$f^{(4)}(x) = \cos x$$

$$f^{(5)}(x) = -\sin x$$

We see that the successive derivatives occur in a cycle of length 4 and, in particular, $f^{(n)}(x) = \cos x$ whenever n is a multiple of 4. Therefore

$$f^{(24)}(x) = \cos x$$

and, differentiating three more times, we have

$$f^{(27)}(x) = \sin x$$

Our main use for the limit in Equation 2 has been to prove the differentiation formula for the sine function. But this limit is also useful in finding certain other trigonometric limits, as the following two examples show.

EXAMPLE 5 Find $\lim_{x \rightarrow 0} \frac{\sin 7x}{4x}$.

SOLUTION In order to apply Equation 2, we first rewrite the function by multiplying and dividing by 7:

Note that $\sin 7x \neq 7 \sin x$.

$$\frac{\sin 7x}{4x} = \frac{7}{4} \left(\frac{\sin 7x}{7x} \right)$$

If we let $\theta = 7x$, then $\theta \rightarrow 0$ as $x \rightarrow 0$, so by Equation 2 we have

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 7x}{4x} &= \frac{7}{4} \lim_{x \rightarrow 0} \left(\frac{\sin 7x}{7x} \right) \\ &= \frac{7}{4} \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \frac{7}{4} \cdot 1 = \frac{7}{4} \end{aligned}$$

V EXAMPLE 6 Calculate $\lim_{x \rightarrow 0} x \cot x$.

SOLUTION Here we divide numerator and denominator by x :

$$\begin{aligned} \lim_{x \rightarrow 0} x \cot x &= \lim_{x \rightarrow 0} \frac{x \cos x}{\sin x} \\ &= \lim_{x \rightarrow 0} \frac{\cos x}{\frac{\sin x}{x}} = \frac{\lim_{x \rightarrow 0} \cos x}{\lim_{x \rightarrow 0} \frac{\sin x}{x}} \\ &= \frac{\cos 0}{1} \quad (\text{by the continuity of cosine and Equation 2}) \\ &= 1 \end{aligned}$$

3.3 Exercises

1–16 Differentiate.

1. $f(x) = 3x^2 - 2 \cos x$

2. $f(x) = \sqrt{x} \sin x$

3. $f(x) = \sin x + \frac{1}{2} \cot x$

4. $y = 2 \sec x - \csc x$

5. $y = \sec \theta \tan \theta$

6. $g(\theta) = e^\theta (\tan \theta - \theta)$

7. $y = c \cos t + t^2 \sin t$

8. $f(t) = \frac{\cot t}{e^t}$

9. $y = \frac{x}{2 - \tan x}$

10. $y = \sin \theta \cos \theta$

11. $f(\theta) = \frac{\sec \theta}{1 + \sec \theta}$

12. $y = \frac{\cos x}{1 - \sin x}$

13. $y = \frac{t \sin t}{1 + t}$

14. $y = \frac{1 - \sec x}{\tan x}$

15. $f(x) = xe^x \csc x$

16. $y = x^2 \sin x \tan x$

17. Prove that $\frac{d}{dx}(\csc x) = -\csc x \cot x$.

18. Prove that $\frac{d}{dx}(\sec x) = \sec x \tan x$.

19. Prove that $\frac{d}{dx}(\cot x) = -\csc^2 x$.

20 Prove, using the definition of derivative, that if $f(x) = \cos x$, then $f'(x) = -\sin x$.

21–24 Find an equation of the tangent line to the curve at the given point.

21. $y = \sec x$, $(\pi/3, 2)$

22. $y = e^x \cos x$, $(0, 1)$

23. $y = \cos x - \sin x$, $(\pi, -1)$

24. $y = x + \tan x$, (π, π)

25. (a) Find an equation of the tangent line to the curve $y = 2x \sin x$ at the point $(\pi/2, \pi)$.

(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

26. (a) Find an equation of the tangent line to the curve $y = 3x + 6 \cos x$ at the point $(\pi/3, \pi + 3)$.

(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

27. (a) If $f(x) = \sec x - x$, find $f'(x)$.(b) Check to see that your answer to part (a) is reasonable by graphing both f and f' for $|x| < \pi/2$.28. (a) If $f(x) = e^x \cos x$, find $f'(x)$ and $f''(x)$.(b) Check to see that your answers to part (a) are reasonable by graphing f , f' , and f'' .29. If $H(\theta) = \theta \sin \theta$, find $H'(\theta)$ and $H''(\theta)$.30. If $f(t) = \csc t$, find $f''(\pi/6)$.

31. (a) Use the Quotient Rule to differentiate the function

$$f(x) = \frac{\tan x - 1}{\sec x}$$

(b) Simplify the expression for $f(x)$ by writing it in terms of $\sin x$ and $\cos x$, and then find $f'(x)$.

(c) Show that your answers to parts (a) and (b) are equivalent.

32. Suppose $f(\pi/3) = 4$ and $f'(\pi/3) = -2$, and let $g(x) = f(x) \sin x$ and $h(x) = (\cos x)/f(x)$. Find

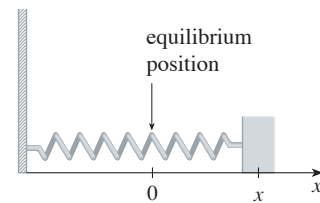
(a) $g'(\pi/3)$

(b) $h'(\pi/3)$

33–34 For what values of x does the graph of f have a horizontal tangent?

33. $f(x) = x + 2 \sin x$

34. $f(x) = e^x \cos x$

35. A mass on a spring vibrates horizontally on a smooth level surface (see the figure). Its equation of motion is $x(t) = 8 \sin t$, where t is in seconds and x in centimeters.(a) Find the velocity and acceleration at time t .(b) Find the position, velocity, and acceleration of the mass at time $t = 2\pi/3$. In what direction is it moving at that time?36. An elastic band is hung on a hook and a mass is hung on the lower end of the band. When the mass is pulled downward and then released, it vibrates vertically. The equation of motion is $s = 2 \cos t + 3 \sin t$, $t \geq 0$, where s is measured in centimeters and t in seconds. (Take the positive direction to be downward.)(a) Find the velocity and acceleration at time t .

(b) Graph the velocity and acceleration functions.

(c) When does the mass pass through the equilibrium position for the first time?

(d) How far from its equilibrium position does the mass travel?

(e) When is the speed the greatest?

37. A ladder 10 ft long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \pi/3$?

38. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin \theta + \cos \theta}$$

where μ is a constant called the *coefficient of friction*.

- (a) Find the rate of change of F with respect to θ .
 (b) When is this rate of change equal to 0?
 (c) If $W = 50$ lb and $\mu = 0.6$, draw the graph of F as a function of θ and use it to locate the value of θ for which $dF/d\theta = 0$. Is the value consistent with your answer to part (b)?



39–48 Find the limit.

39. $\lim_{x \rightarrow 0} \frac{\sin 3x}{x}$

40. $\lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 6x}$

41. $\lim_{t \rightarrow 0} \frac{\tan 6t}{\sin 2t}$

42. $\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\sin \theta}$

43. $\lim_{x \rightarrow 0} \frac{\sin 3x}{5x^3 - 4x}$

44. $\lim_{x \rightarrow 0} \frac{\sin 3x \sin 5x}{x^2}$

45. $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta + \tan \theta}$

46. $\lim_{x \rightarrow 0} \frac{\sin(x^2)}{x}$

47. $\lim_{x \rightarrow \pi/4} \frac{1 - \tan x}{\sin x - \cos x}$

48. $\lim_{x \rightarrow 1} \frac{\sin(x-1)}{x^2 + x - 2}$

49–50 Find the given derivative by finding the first few derivatives and observing the pattern that occurs.

49. $\frac{d^{99}}{dx^{99}}(\sin x)$

50. $\frac{d^{35}}{dx^{35}}(x \sin x)$

51. Find constants A and B such that the function $y = A \sin x + B \cos x$ satisfies the differential equation $y'' + y' - 2y = \sin x$.

52. (a) Evaluate $\lim_{x \rightarrow \infty} x \sin \frac{1}{x}$.

(b) Evaluate $\lim_{x \rightarrow 0} x \sin \frac{1}{x}$.



- (c) Illustrate parts (a) and (b) by graphing $y = x \sin(1/x)$.

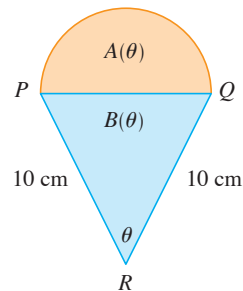
53. Differentiate each trigonometric identity to obtain a new (or familiar) identity.

(a) $\tan x = \frac{\sin x}{\cos x}$ (b) $\sec x = \frac{1}{\cos x}$

(c) $\sin x + \cos x = \frac{1 + \cot x}{\csc x}$

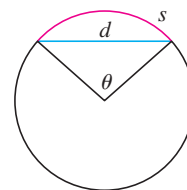
54. A semicircle with diameter PQ sits on an isosceles triangle PQR to form a region shaped like a two-dimensional ice-cream cone, as shown in the figure. If $A(\theta)$ is the area of the semicircle and $B(\theta)$ is the area of the triangle, find

$$\lim_{\theta \rightarrow 0^+} \frac{A(\theta)}{B(\theta)}$$



55. The figure shows a circular arc of length s and a chord of length d , both subtended by a central angle θ . Find

$$\lim_{\theta \rightarrow 0^+} \frac{s}{d}$$



56. Let $f(x) = \frac{x}{\sqrt{1 - \cos 2x}}$.

- (a) Graph f . What type of discontinuity does it appear to have at 0?
 (b) Calculate the left and right limits of f at 0. Do these values confirm your answer to part (a)?

3.4 The Chain Rule

Suppose you are asked to differentiate the function

$$F(x) = \sqrt{x^2 + 1}$$

The differentiation formulas you learned in the previous sections of this chapter do not enable you to calculate $F'(x)$.

See Section 1.3 for a review of composite functions.

Observe that F is a composite function. In fact, if we let $y = f(u) = \sqrt{u}$ and let $u = g(x) = x^2 + 1$, then we can write $y = F(x) = f(g(x))$, that is, $F = f \circ g$. We know how to differentiate both f and g , so it would be useful to have a rule that tells us how to find the derivative of $F = f \circ g$ in terms of the derivatives of f and g .

It turns out that the derivative of the composite function $f \circ g$ is the product of the derivatives of f and g . This fact is one of the most important of the differentiation rules and is called the *Chain Rule*. It seems plausible if we interpret derivatives as rates of change. Regard du/dx as the rate of change of u with respect to x , dy/du as the rate of change of y with respect to u , and dy/dx as the rate of change of y with respect to x . If u changes twice as fast as x and y changes three times as fast as u , then it seems reasonable that y changes six times as fast as x , and so we expect that

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

James Gregory

The first person to formulate the Chain Rule was the Scottish mathematician James Gregory (1638–1675), who also designed the first practical reflecting telescope. Gregory discovered the basic ideas of calculus at about the same time as Newton. He became the first Professor of Mathematics at the University of St. Andrews and later held the same position at the University of Edinburgh. But one year after accepting that position he died at the age of 36.

COMMENTS ON THE PROOF OF THE CHAIN RULE Let Δu be the change in u corresponding to a change of Δx in x , that is,

$$\Delta u = g(x + \Delta x) - g(x)$$

Then the corresponding change in y is

$$\Delta y = f(u + \Delta u) - f(u)$$

It is tempting to write

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} \\ &= \lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} \quad (\text{Note that } \Delta u \rightarrow 0 \text{ as } \Delta x \rightarrow 0 \\ &\quad \text{since } g \text{ is continuous.}) \\ &= \frac{dy}{du} \frac{du}{dx} \end{aligned}$$

The only flaw in this reasoning is that in $\boxed{1}$ it might happen that $\Delta u = 0$ (even when $\Delta x \neq 0$) and, of course, we can't divide by 0. Nonetheless, this reasoning does at least *suggest* that the Chain Rule is true. A full proof of the Chain Rule is given at the end of this section.

The Chain Rule can be written either in the prime notation

$$\boxed{2} \quad (f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

or, if $y = f(u)$ and $u = g(x)$, in Leibniz notation:

$$\boxed{3} \quad \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

Equation 3 is easy to remember because if dy/du and du/dx were quotients, then we could cancel du . Remember, however, that du has not been defined and du/dx should not be thought of as an actual quotient.

EXAMPLE 1 Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

SOLUTION 1 (using Equation 2): At the beginning of this section we expressed F as $F(x) = (f \circ g)(x) = f(g(x))$ where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$. Since

$$f'(u) = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} \quad \text{and} \quad g'(x) = 2x$$

we have

$$\begin{aligned} F'(x) &= f'(g(x)) \cdot g'(x) \\ &= \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x = \frac{x}{\sqrt{x^2 + 1}} \end{aligned}$$

SOLUTION 2 (using Equation 3): If we let $u = x^2 + 1$ and $y = \sqrt{u}$, then

$$F'(x) = \frac{dy}{du} \frac{du}{dx} = \frac{1}{2\sqrt{u}} (2x) = \frac{1}{2\sqrt{x^2 + 1}} (2x) = \frac{x}{\sqrt{x^2 + 1}}$$

When using Formula 3 we should bear in mind that dy/dx refers to the derivative of y when y is considered as a function of x (called the *derivative of y with respect to x*), whereas dy/du refers to the derivative of y when considered as a function of u (the derivative of y with respect to u). For instance, in Example 1, y can be considered as a function of x ($y = \sqrt{x^2 + 1}$) and also as a function of u ($y = \sqrt{u}$). Note that

$$\frac{dy}{dx} = F'(x) = \frac{x}{\sqrt{x^2 + 1}} \quad \text{whereas} \quad \frac{dy}{du} = f'(u) = \frac{1}{2\sqrt{u}}$$

NOTE In using the Chain Rule we work from the outside to the inside. Formula 2 says that we *differentiate the outer function f [at the inner function $g(x)$] and then we multiply by the derivative of the inner function.*

$$\frac{d}{dx} \underbrace{f}_{\text{outer function}} \underbrace{(g(x))}_{\substack{\text{evaluated} \\ \text{at inner} \\ \text{function}}} = \underbrace{f'}_{\substack{\text{derivative} \\ \text{of outer} \\ \text{function}}} \underbrace{(g(x))}_{\substack{\text{evaluated} \\ \text{at inner} \\ \text{function}}} \cdot \underbrace{g'(x)}_{\substack{\text{derivative} \\ \text{of inner} \\ \text{function}}}$$

V EXAMPLE 2 Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$.

SOLUTION

(a) If $y = \sin(x^2)$, then the outer function is the sine function and the inner function is the squaring function, so the Chain Rule gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \underbrace{\sin}_{\text{outer function}} \underbrace{(x^2)}_{\text{evaluated at inner function}} = \underbrace{\cos}_{\text{derivative of outer function}} \underbrace{(x^2)}_{\text{evaluated at inner function}} \cdot \underbrace{2x}_{\text{derivative of inner function}} \\ &= 2x \cos(x^2) \end{aligned}$$

(b) Note that $\sin^2 x = (\sin x)^2$. Here the outer function is the squaring function and the inner function is the sine function. So

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \underbrace{(\sin x)^2}_{\text{inner function}} = \underbrace{2}_{\text{derivative of outer function}} \cdot \underbrace{(\sin x)}_{\text{evaluated at inner function}} \cdot \underbrace{\cos x}_{\text{derivative of inner function}} \end{aligned}$$

See Reference Page 2 or Appendix D.

The answer can be left as $2 \sin x \cos x$ or written as $\sin 2x$ (by a trigonometric identity known as the double-angle formula).

In Example 2(a) we combined the Chain Rule with the rule for differentiating the sine function. In general, if $y = \sin u$, where u is a differentiable function of x , then, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \cos u \frac{du}{dx}$$

Thus
$$\frac{d}{dx} (\sin u) = \cos u \frac{du}{dx}$$

In a similar fashion, all of the formulas for differentiating trigonometric functions can be combined with the Chain Rule.

Let's make explicit the special case of the Chain Rule where the outer function f is a power function. If $y = [g(x)]^n$, then we can write $y = f(u) = u^n$ where $u = g(x)$. By using the Chain Rule and then the Power Rule, we get

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = nu^{n-1} \frac{du}{dx} = n[g(x)]^{n-1} g'(x)$$

4 The Power Rule Combined with the Chain Rule If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx} (u^n) = nu^{n-1} \frac{du}{dx}$$

Alternatively,
$$\frac{d}{dx} [g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Notice that the derivative in Example 1 could be calculated by taking $n = \frac{1}{2}$ in Rule 4.

EXAMPLE 3 Differentiate $y = (x^3 - 1)^{100}$.**SOLUTION** Taking $u = g(x) = x^3 - 1$ and $n = 100$ in [4], we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} (x^3 - 1)^{100} = 100(x^3 - 1)^{99} \frac{d}{dx} (x^3 - 1) \\ &= 100(x^3 - 1)^{99} \cdot 3x^2 = 300x^2(x^3 - 1)^{99}\end{aligned}$$

V EXAMPLE 4 Find $f'(x)$ if $f(x) = \frac{1}{\sqrt[3]{x^2 + x + 1}}$.**SOLUTION** First rewrite f : $f(x) = (x^2 + x + 1)^{-1/3}$

Thus

$$\begin{aligned}f'(x) &= -\frac{1}{3}(x^2 + x + 1)^{-4/3} \frac{d}{dx} (x^2 + x + 1) \\ &= -\frac{1}{3}(x^2 + x + 1)^{-4/3}(2x + 1)\end{aligned}$$

EXAMPLE 5 Find the derivative of the function

$$g(t) = \left(\frac{t-2}{2t+1} \right)^9$$

SOLUTION Combining the Power Rule, Chain Rule, and Quotient Rule, we get

$$\begin{aligned}g'(t) &= 9 \left(\frac{t-2}{2t+1} \right)^8 \frac{d}{dt} \left(\frac{t-2}{2t+1} \right) \\ &= 9 \left(\frac{t-2}{2t+1} \right)^8 \frac{(2t+1) \cdot 1 - 2(t-2)}{(2t+1)^2} = \frac{45(t-2)^8}{(2t+1)^{10}}\end{aligned}$$

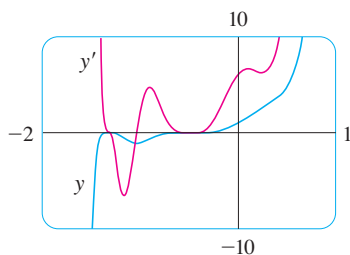
EXAMPLE 6 Differentiate $y = (2x + 1)^5(x^3 - x + 1)^4$.**SOLUTION** In this example we must use the Product Rule before using the Chain Rule:

$$\begin{aligned}\frac{dy}{dx} &= (2x + 1)^5 \frac{d}{dx} (x^3 - x + 1)^4 + (x^3 - x + 1)^4 \frac{d}{dx} (2x + 1)^5 \\ &= (2x + 1)^5 \cdot 4(x^3 - x + 1)^3 \frac{d}{dx} (x^3 - x + 1) \\ &\quad + (x^3 - x + 1)^4 \cdot 5(2x + 1)^4 \frac{d}{dx} (2x + 1) \\ &= 4(2x + 1)^5(x^3 - x + 1)^3(3x^2 - 1) + 5(x^3 - x + 1)^4(2x + 1)^4 \cdot 2\end{aligned}$$

Noticing that each term has the common factor $2(2x + 1)^4(x^3 - x + 1)^3$, we could factor it out and write the answer as

$$\frac{dy}{dx} = 2(2x + 1)^4(x^3 - x + 1)^3(17x^3 + 6x^2 - 9x + 3)$$

The graphs of the functions y and y' in Example 6 are shown in Figure 1. Notice that y' is large when y increases rapidly and $y' = 0$ when y has a horizontal tangent. So our answer appears to be reasonable.

**FIGURE 1****EXAMPLE 7** Differentiate $y = e^{\sin x}$.

SOLUTION Here the inner function is $g(x) = \sin x$ and the outer function is the exponential function $f(x) = e^x$. So, by the Chain Rule,

More generally, the Chain Rule gives

$$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\sin x}) = e^{\sin x} \frac{d}{dx}(\sin x) = e^{\sin x} \cos x$$

We can use the Chain Rule to differentiate an exponential function with any base $a > 0$. Recall from Section 1.6 that $a = e^{\ln a}$. So

$$a^x = (e^{\ln a})^x = e^{(\ln a)x}$$

and the Chain Rule gives

$$\begin{aligned} \frac{d}{dx}(a^x) &= \frac{d}{dx}(e^{(\ln a)x}) = e^{(\ln a)x} \frac{d}{dx}(\ln a)x \\ &= e^{(\ln a)x} \cdot \ln a = a^x \ln a \end{aligned}$$

because $\ln a$ is a constant. So we have the formula

Don't confuse Formula 5 (where x is the *exponent*) with the Power Rule (where x is the *base*):

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

5

$$\frac{d}{dx}(a^x) = a^x \ln a$$

In particular, if $a = 2$, we get

6

$$\frac{d}{dx}(2^x) = 2^x \ln 2$$

In Section 3.1 we gave the estimate

$$\frac{d}{dx}(2^x) \approx (0.69)2^x$$

This is consistent with the exact formula [6] because $\ln 2 \approx 0.693147$.

The reason for the name “Chain Rule” becomes clear when we make a longer chain by adding another link. Suppose that $y = f(u)$, $u = g(x)$, and $x = h(t)$, where f , g , and h are differentiable functions. Then, to compute the derivative of y with respect to t , we use the Chain Rule twice:

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = \frac{dy}{du} \frac{du}{dx} \frac{dx}{dt}$$

V EXAMPLE 8 If $f(x) = \sin(\cos(\tan x))$, then

$$\begin{aligned} f'(x) &= \cos(\cos(\tan x)) \frac{d}{dx} \cos(\tan x) \\ &= \cos(\cos(\tan x)) [-\sin(\tan x)] \frac{d}{dx}(\tan x) \\ &= -\cos(\cos(\tan x)) \sin(\tan x) \sec^2 x \end{aligned}$$

Notice that we used the Chain Rule twice.

EXAMPLE 9 Differentiate $y = e^{\sec 3\theta}$.

SOLUTION The outer function is the exponential function, the middle function is the secant function, and the inner function is the tripling function. So we have

$$\begin{aligned}\frac{dy}{d\theta} &= e^{\sec 3\theta} \frac{d}{d\theta} (\sec 3\theta) \\ &= e^{\sec 3\theta} \sec 3\theta \tan 3\theta \frac{d}{d\theta} (3\theta) \\ &= 3e^{\sec 3\theta} \sec 3\theta \tan 3\theta\end{aligned}$$

How to Prove the Chain Rule

Recall that if $y = f(x)$ and x changes from a to $a + \Delta x$, we define the increment of y as

$$\Delta y = f(a + \Delta x) - f(a)$$

According to the definition of a derivative, we have

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = f'(a)$$

So if we denote by ε the difference between the difference quotient and the derivative, we obtain

$$\lim_{\Delta x \rightarrow 0} \varepsilon = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta x} - f'(a) \right) = f'(a) - f'(a) = 0$$

But
$$\varepsilon = \frac{\Delta y}{\Delta x} - f'(a) \quad \Rightarrow \quad \Delta y = f'(a) \Delta x + \varepsilon \Delta x$$

If we define ε to be 0 when $\Delta x = 0$, then ε becomes a continuous function of Δx . Thus, for a differentiable function f , we can write

$$\boxed{7} \quad \Delta y = f'(a) \Delta x + \varepsilon \Delta x \quad \text{where } \varepsilon \rightarrow 0 \text{ as } \Delta x \rightarrow 0$$

and ε is a continuous function of Δx . This property of differentiable functions is what enables us to prove the Chain Rule.

PROOF OF THE CHAIN RULE Suppose $u = g(x)$ is differentiable at a and $y = f(u)$ is differentiable at $b = g(a)$. If Δx is an increment in x and Δu and Δy are the corresponding increments in u and y , then we can use Equation 7 to write

$$\boxed{8} \quad \Delta u = g'(a) \Delta x + \varepsilon_1 \Delta x = [g'(a) + \varepsilon_1] \Delta x$$

where $\varepsilon_1 \rightarrow 0$ as $\Delta x \rightarrow 0$. Similarly

$$\boxed{9} \quad \Delta y = f'(b) \Delta u + \varepsilon_2 \Delta u = [f'(b) + \varepsilon_2] \Delta u$$

where $\varepsilon_2 \rightarrow 0$ as $\Delta u \rightarrow 0$. If we now substitute the expression for Δu from Equation 8 into Equation 9, we get

$$\Delta y = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1] \Delta x$$

$$\text{so} \quad \frac{\Delta y}{\Delta x} = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]$$

As $\Delta x \rightarrow 0$, Equation 8 shows that $\Delta u \rightarrow 0$. So both $\varepsilon_1 \rightarrow 0$ and $\varepsilon_2 \rightarrow 0$ as $\Delta x \rightarrow 0$. Therefore

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1] \\ &= f'(b)g'(a) = f'(g(a))g'(a) \end{aligned}$$

This proves the Chain Rule.

3.4 Exercises

1–6 Write the composite function in the form $f(g(x))$. [Identify the inner function $u = g(x)$ and the outer function $y = f(u)$.] Then find the derivative dy/dx .

1. $y = \sqrt[3]{1 + 4x}$

2. $y = (2x^3 + 5)^4$

3. $y = \tan \pi x$

4. $y = \sin(\cot x)$

5. $y = e^{\sqrt{x}}$

6. $y = \sqrt{2 - e^x}$

7–46 Find the derivative of the function.

7. $F(x) = (x^4 + 3x^2 - 2)^5$

8. $F(x) = (4x - x^2)^{100}$

9. $F(x) = \sqrt{1 - 2x}$

10. $f(x) = \frac{1}{(1 + \sec x)^2}$

11. $f(z) = \frac{1}{z^2 + 1}$

12. $f(t) = \sin(e^t) + e^{\sin t}$

13. $y = \cos(a^3 + x^3)$

14. $y = a^3 + \cos^3 x$

15. $y = xe^{-kx}$

16. $y = e^{-2t} \cos 4t$

17. $f(x) = (2x - 3)^4(x^2 + x + 1)^5$

18. $g(x) = (x^2 + 1)^3(x^2 + 2)^6$

19. $h(t) = (t + 1)^{2/3}(2t^2 - 1)^3$

20. $F(t) = (3t - 1)^4(2t + 1)^{-3}$

21. $y = \left(\frac{x^2 + 1}{x^2 - 1} \right)^3$

22. $f(s) = \sqrt{\frac{s^2 + 1}{s^2 + 4}}$

23. $y = \sqrt{1 + 2e^{3x}}$

24. $y = 10^{1-x^2}$

25. $y = 5^{-1/x}$

26. $G(y) = \frac{(y - 1)^4}{(y^2 + 2y)^5}$

27. $y = \frac{r}{\sqrt{r^2 + 1}}$

28. $y = \frac{e^u - e^{-u}}{e^u + e^{-u}}$

29. $F(t) = e^{t \sin 2t}$

30. $F(v) = \left(\frac{v}{v^3 + 1} \right)^6$

31. $y = \sin(\tan 2x)$

32. $y = \sec^2(m\theta)$

33. $y = 2^{\sin \pi x}$

34. $y = x^2 e^{-1/x}$

35. $y = \cos\left(\frac{1 - e^{2x}}{1 + e^{2x}}\right)$

36. $y = \sqrt{1 + xe^{-2x}}$

37. $y = \cot^2(\sin \theta)$

38. $y = e^{k \tan \sqrt{x}}$

39. $f(t) = \tan(e^t) + e^{\tan t}$

40. $y = \sin(\sin(\sin x))$

41. $f(t) = \sin^2(e^{\sin t})$

42. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

43. $g(x) = (2ra^{rx} + n)^p$

44. $y = 2^{3x^2}$

45. $y = \cos \sqrt{\sin(\tan \pi x)}$

46. $y = [x + (x + \sin^2 x)^3]^4$

47–50 Find y' and y'' .

47. $y = \cos(x^2)$

48. $y = \cos^2 x$

49. $y = e^{\alpha x} \sin \beta x$

50. $y = e^{e^x}$

51–54 Find an equation of the tangent line to the curve at the given point.

51. $y = (1 + 2x)^{10}, \quad (0, 1)$

52. $y = \sqrt{1 + x^3}, \quad (2, 3)$

53. $y = \sin(\sin x), \quad (\pi, 0)$

54. $y = \sin x + \sin^2 x, \quad (0, 0)$

55. (a) Find an equation of the tangent line to the curve $y = 2/(1 + e^{-x})$ at the point $(0, 1)$.



(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

56. (a) The curve $y = |x|/\sqrt{2 - x^2}$ is called a *bullet-nose curve*. Find an equation of the tangent line to this curve at the point $(1, 1)$.



(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

57. (a) If $f(x) = x\sqrt{2 - x^2}$, find $f'(x)$.



(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f' .




Graphing calculator or computer required



Computer algebra system required

1. Homework Hints available at stewartcalculus.com

-  **58.** The function $f(x) = \sin(x + \sin 2x)$, $0 \leq x \leq \pi$, arises in applications to frequency modulation (FM) synthesis.
- (a) Use a graph of f produced by a graphing device to make a rough sketch of the graph of f' .
- (b) Calculate $f'(x)$ and use this expression, with a graphing device, to graph f' . Compare with your sketch in part (a).

- 59.** Find all points on the graph of the function $f(x) = 2 \sin x + \sin^2 x$ at which the tangent line is horizontal.

- 60.** Find the x -coordinates of all points on the curve $y = \sin 2x - 2 \sin x$ at which the tangent line is horizontal.

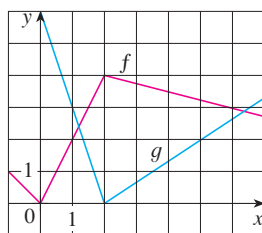
- 61.** If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

- 62.** If $h(x) = \sqrt{4 + 3f(x)}$, where $f(1) = 7$ and $f'(1) = 4$, find $h'(1)$.

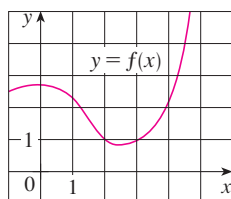
- 63.** A table of values for f , g , f' , and g' is given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

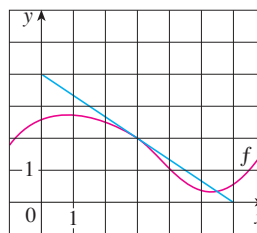
- (a) If $h(x) = f(g(x))$, find $h'(1)$.
- (b) If $H(x) = g(f(x))$, find $H'(1)$.
- 64.** Let f and g be the functions in Exercise 63.
- (a) If $F(x) = f(f(x))$, find $F'(2)$.
- (b) If $G(x) = g(g(x))$, find $G'(3)$.
- 65.** If f and g are the functions whose graphs are shown, let $u(x) = f(g(x))$, $v(x) = g(f(x))$, and $w(x) = g(g(x))$. Find each derivative, if it exists. If it does not exist, explain why.
- (a) $u'(1)$ (b) $v'(1)$ (c) $w'(1)$



- 66.** If f is the function whose graph is shown, let $h(x) = f(f(x))$ and $g(x) = f(x^2)$. Use the graph of f to estimate the value of each derivative.
- (a) $h'(2)$ (b) $g'(2)$



- 67.** If $g(x) = \sqrt{f(x)}$, where the graph of f is shown, evaluate $g'(3)$.



- 68.** Suppose f is differentiable on \mathbb{R} and α is a real number. Let $F(x) = f(x^\alpha)$ and $G(x) = [f(x)]^\alpha$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.
- 69.** Suppose f is differentiable on \mathbb{R} . Let $F(x) = f(e^x)$ and $G(x) = e^{f(x)}$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.
- 70.** Let $g(x) = e^{cx} + f(x)$ and $h(x) = e^{kx}f(x)$, where $f(0) = 3$, $f'(0) = 5$, and $f''(0) = -2$.
- (a) Find $g'(0)$ and $g''(0)$ in terms of c .
- (b) In terms of k , find an equation of the tangent line to the graph of h at the point where $x = 0$.
- 71.** Let $r(x) = f(g(h(x)))$, where $h(1) = 2$, $g(2) = 3$, $h'(1) = 4$, $g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.
- 72.** If g is a twice differentiable function and $f(x) = xg(x^2)$, find f'' in terms of g , g' , and g'' .
- 73.** If $F(x) = f(3f(4f(x)))$, where $f(0) = 0$ and $f'(0) = 2$, find $F'(0)$.
- 74.** If $F(x) = f(xf(xf(x)))$, where $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$, $f'(2) = 5$, and $f'(3) = 6$, find $F'(1)$.
- 75.** Show that the function $y = e^{2x}(A \cos 3x + B \sin 3x)$ satisfies the differential equation $y'' - 4y' + 13y = 0$.
- 76.** For what values of r does the function $y = e^{rx}$ satisfy the differential equation $y'' - 4y' + y = 0$?
- 77.** Find the 50th derivative of $y = \cos 2x$.
- 78.** Find the 1000th derivative of $f(x) = xe^{-x}$.
- 79.** The displacement of a particle on a vibrating string is given by the equation $s(t) = 10 + \frac{1}{4} \sin(10\pi t)$ where s is measured in centimeters and t in seconds. Find the velocity of the particle after t seconds.

- 80.** If the equation of motion of a particle is given by $s = A \cos(\omega t + \delta)$, the particle is said to undergo *simple harmonic motion*.
- (a) Find the velocity of the particle at time t .
- (b) When is the velocity 0?
- 81.** A Cepheid variable star is a star whose brightness alternately increases and decreases. The most easily visible such star is Delta Cephei, for which the interval between times of maximum brightness is 5.4 days. The average brightness of this star is 4.0 and its brightness changes by ± 0.35 . In view of these data, the brightness of Delta Cephei at time t , where t is mea-

sured in days, has been modeled by the function


$$B(t) = 4.0 + 0.35 \sin\left(\frac{2\pi t}{5.4}\right)$$

- Find the rate of change of the brightness after t days.
- Find, correct to two decimal places, the rate of increase after one day.

- 82.** In Example 4 in Section 1.3 we arrived at a model for the length of daylight (in hours) in Philadelphia on the t th day of the year:

$$L(t) = 12 + 2.8 \sin\left[\frac{2\pi}{365}(t - 80)\right]$$

Use this model to compare how the number of hours of daylight is increasing in Philadelphia on March 21 and May 21.

-  **83.** The motion of a spring that is subject to a frictional force or a damping force (such as a shock absorber in a car) is often modeled by the product of an exponential function and a sine or cosine function. Suppose the equation of motion of a point on such a spring is


$$s(t) = 2e^{-1.5t} \sin 2\pi t$$

where s is measured in centimeters and t in seconds. Find the velocity after t seconds and graph both the position and velocity functions for $0 \leq t \leq 2$.

- 84.** Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that knows the rumor at time t and a and k are positive constants. [In Section 9.4 we will see that this is a reasonable equation for $p(t)$.]


- Find $\lim_{t \rightarrow \infty} p(t)$.
 - Find the rate of spread of the rumor.
-  **(c)** Graph p for the case $a = 10$, $k = 0.5$ with t measured in hours. Use the graph to estimate how long it will take for 80% of the population to hear the rumor.

- 85.** A particle moves along a straight line with displacement $s(t)$, velocity $v(t)$, and acceleration $a(t)$. Show that

$$a(t) = v(t) \frac{dv}{ds}$$

Explain the difference between the meanings of the derivatives dv/dt and dv/ds .

- 86.** Air is being pumped into a spherical weather balloon. At any time t , the volume of the balloon is $V(t)$ and its radius is $r(t)$.
- What do the derivatives dV/dr and dV/dt represent?
 - Express dV/dt in terms of dr/dt .

-  **87.** The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off.

The following data describe the charge Q remaining on the capacitor (measured in microcoulombs, μC) at time t (measured in seconds).


t	0.00	0.02	0.04	0.06	0.08	0.10
Q	100.00	81.87	67.03	54.88	44.93	36.76

- Use a graphing calculator or computer to find an exponential model for the charge.
- The derivative $Q'(t)$ represents the electric current (measured in microamperes, μA) flowing from the capacitor to the flash bulb. Use part (a) to estimate the current when $t = 0.04$ s. Compare with the result of Example 2 in Section 2.1.


-  **88.** The table gives the US population from 1790 to 1860.

Year	Population	Year	Population
1790	3,929,000	1830	12,861,000
1800	5,308,000	1840	17,063,000
1810	7,240,000	1850	23,192,000
1820	9,639,000	1860	31,443,000

- Use a graphing calculator or computer to fit an exponential function to the data. Graph the data points and the exponential model. How good is the fit?
- Estimate the rates of population growth in 1800 and 1850 by averaging slopes of secant lines.
- Use the exponential model in part (a) to estimate the rates of growth in 1800 and 1850. Compare these estimates with the ones in part (b).
- Use the exponential model to predict the population in 1870. Compare with the actual population of 38,558,000. Can you explain the discrepancy?

-  **89.** Computer algebra systems have commands that differentiate functions, but the form of the answer may not be convenient and so further commands may be necessary to simplify the answer.

- Use a CAS to find the derivative in Example 5 and compare with the answer in that example. Then use the simplify command and compare again.
- Use a CAS to find the derivative in Example 6. What happens if you use the simplify command? What happens if you use the factor command? Which form of the answer would be best for locating horizontal tangents?

-  **90.** (a) Use a CAS to differentiate the function

$$f(x) = \sqrt{\frac{x^4 - x + 1}{x^4 + x + 1}}$$

and to simplify the result.

- Where does the graph of f have horizontal tangents?
- Graph f and f' on the same screen. Are the graphs consistent with your answer to part (b)?

91. Use the Chain Rule to prove the following.
- The derivative of an even function is an odd function.
 - The derivative of an odd function is an even function.

92. Use the Chain Rule and the Product Rule to give an alternative proof of the Quotient Rule.

[Hint: Write $f(x)/g(x) = f(x)[g(x)]^{-1}$.]

93. (a) If n is a positive integer, prove that

$$\frac{d}{dx}(\sin^n x \cos nx) = n \sin^{n-1} x \cos(n+1)x$$

- (b) Find a formula for the derivative of $y = \cos^n x \cos nx$ that is similar to the one in part (a).

94. Suppose $y = f(x)$ is a curve that always lies above the x -axis and never has a horizontal tangent, where f is differentiable everywhere. For what value of y is the rate of change of y^5 with respect to x eighty times the rate of change of y with respect to x ?

95. Use the Chain Rule to show that if θ is measured in degrees, then

$$\frac{d}{d\theta}(\sin \theta) = \frac{\pi}{180} \cos \theta$$

(This gives one reason for the convention that radian measure is always used when dealing with trigonometric functions in calculus: The differentiation formulas would not be as simple if we used degree measure.)

96. (a) Write $|x| = \sqrt{x^2}$ and use the Chain Rule to show that

$$\frac{d}{dx}|x| = \frac{x}{|x|}$$

- If $f(x) = |\sin x|$, find $f'(x)$ and sketch the graphs of f and f' . Where is f not differentiable?
- If $g(x) = \sin |x|$, find $g'(x)$ and sketch the graphs of g and g' . Where is g not differentiable?

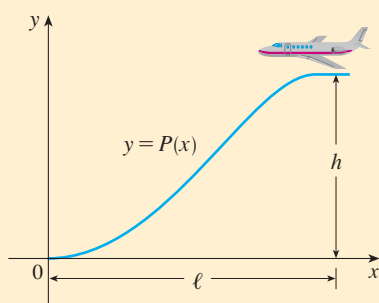
97. If $y = f(u)$ and $u = g(x)$, where f and g are twice differentiable functions, show that

$$\frac{d^2 y}{dx^2} = \frac{d^2 y}{du^2} \left(\frac{du}{dx} \right)^2 + \frac{dy}{du} \frac{d^2 u}{dx^2}$$

98. If $y = f(u)$ and $u = g(x)$, where f and g possess third derivatives, find a formula for $d^3 y/dx^3$ similar to the one given in Exercise 97.

APPLIED PROJECT

WHERE SHOULD A PILOT START DESCENT?



An approach path for an aircraft landing is shown in the figure and satisfies the following conditions:

- The cruising altitude is h when descent starts at a horizontal distance ℓ from touchdown at the origin.
- The pilot must maintain a constant horizontal speed v throughout descent.
- The absolute value of the vertical acceleration should not exceed a constant k (which is much less than the acceleration due to gravity).

- Find a cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ that satisfies condition (i) by imposing suitable conditions on $P(x)$ and $P'(x)$ at the start of descent and at touchdown.
- Use conditions (ii) and (iii) to show that

$$\frac{6hv^2}{\ell^2} \leq k$$

- Suppose that an airline decides not to allow vertical acceleration of a plane to exceed $k = 860 \text{ mi/h}^2$. If the cruising altitude of a plane is 35,000 ft and the speed is 300 mi/h, how far away from the airport should the pilot start descent?



4. Graph the approach path if the conditions stated in Problem 3 are satisfied.



Graphing calculator or computer required

3.5 Implicit Differentiation

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable—for example,

$$y = \sqrt{x^3 + 1} \quad \text{or} \quad y = x \sin x$$

or, in general, $y = f(x)$. Some functions, however, are defined implicitly by a relation between x and y such as

$$\boxed{1} \quad x^2 + y^2 = 25$$

or

$$\boxed{2} \quad x^3 + y^3 = 6xy$$

In some cases it is possible to solve such an equation for y as an explicit function (or several functions) of x . For instance, if we solve Equation 1 for y , we get $y = \pm\sqrt{25 - x^2}$, so two of the functions determined by the implicit Equation 1 are $f(x) = \sqrt{25 - x^2}$ and $g(x) = -\sqrt{25 - x^2}$. The graphs of f and g are the upper and lower semicircles of the circle $x^2 + y^2 = 25$. (See Figure 1.)

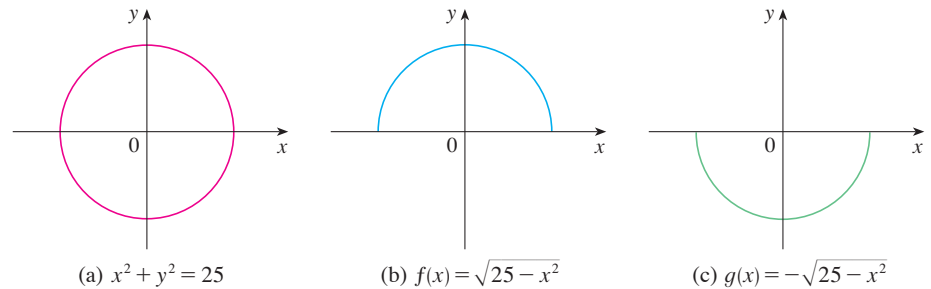


FIGURE 1

It's not easy to solve Equation 2 for y explicitly as a function of x by hand. (A computer algebra system has no trouble, but the expressions it obtains are very complicated.) Nonetheless, $\boxed{2}$ is the equation of a curve called the **folium of Descartes** shown in Figure 2 and it implicitly defines y as several functions of x . The graphs of three such functions are shown in Figure 3. When we say that f is a function defined implicitly by Equation 2, we mean that the equation

$$x^3 + [f(x)]^3 = 6xf(x)$$

is true for all values of x in the domain of f .

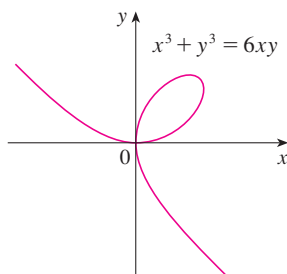


FIGURE 2 The folium of Descartes

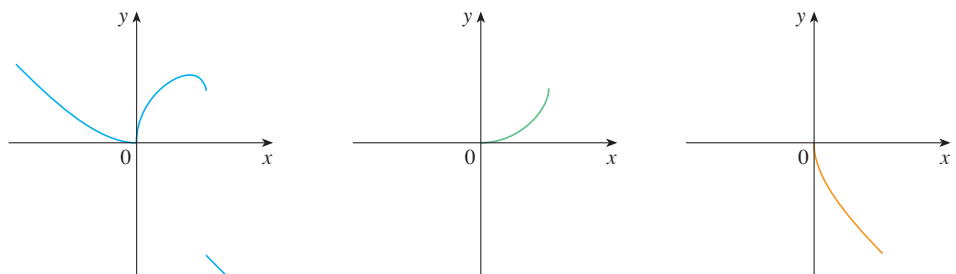


FIGURE 3 Graphs of three functions defined by the folium of Descartes

Fortunately, we don't need to solve an equation for y in terms of x in order to find the derivative of y . Instead we can use the method of **implicit differentiation**. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y' . In the examples and exercises of this section it is always assumed that the given equation determines y implicitly as a differentiable function of x so that the method of implicit differentiation can be applied.

V EXAMPLE 1

- (a) If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.
 (b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$.

SOLUTION 1

- (a) Differentiate both sides of the equation $x^2 + y^2 = 25$:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Remembering that y is a function of x and using the Chain Rule, we have

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \frac{dy}{dx} = 2y \frac{dy}{dx}$$

Thus
$$2x + 2y \frac{dy}{dx} = 0$$

Now we solve this equation for dy/dx :

$$\frac{dy}{dx} = -\frac{x}{y}$$

- (b) At the point $(3, 4)$ we have $x = 3$ and $y = 4$, so

$$\frac{dy}{dx} = -\frac{3}{4}$$

An equation of the tangent to the circle at $(3, 4)$ is therefore

$$y - 4 = -\frac{3}{4}(x - 3) \quad \text{or} \quad 3x + 4y = 25$$

SOLUTION 2

- (b) Solving the equation $x^2 + y^2 = 25$, we get $y = \pm\sqrt{25 - x^2}$. The point $(3, 4)$ lies on the upper semicircle $y = \sqrt{25 - x^2}$ and so we consider the function $f(x) = \sqrt{25 - x^2}$. Differentiating f using the Chain Rule, we have

$$\begin{aligned} f'(x) &= \frac{1}{2}(25 - x^2)^{-1/2} \frac{d}{dx}(25 - x^2) \\ &= \frac{1}{2}(25 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{25 - x^2}} \end{aligned}$$

Example 1 illustrates that even when it is possible to solve an equation explicitly for y in terms of x , it may be easier to use implicit differentiation.

So
$$f'(3) = -\frac{3}{\sqrt{25-3^2}} = -\frac{3}{4}$$

and, as in Solution 1, an equation of the tangent is $3x + 4y = 25$.

NOTE 1 The expression $dy/dx = -x/y$ in Solution 1 gives the derivative in terms of both x and y . It is correct no matter which function y is determined by the given equation. For instance, for $y = f(x) = \sqrt{25 - x^2}$ we have

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{\sqrt{25 - x^2}}$$

whereas for $y = g(x) = -\sqrt{25 - x^2}$ we have

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{-\sqrt{25 - x^2}} = \frac{x}{\sqrt{25 - x^2}}$$

EXAMPLE 2

- Find y' if $x^3 + y^3 = 6xy$.
- Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3, 3)$.
- At what point in the first quadrant is the tangent line horizontal?

SOLUTION

(a) Differentiating both sides of $x^3 + y^3 = 6xy$ with respect to x , regarding y as a function of x , and using the Chain Rule on the term y^3 and the Product Rule on the term $6xy$, we get

$$3x^2 + 3y^2y' = 6xy' + 6y$$

or

$$x^2 + y^2y' = 2xy' + 2y$$

We now solve for y' :

$$y^2y' - 2xy' = 2y - x^2$$

$$(y^2 - 2x)y' = 2y - x^2$$

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

(b) When $x = y = 3$,

$$y' = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = -1$$

and a glance at Figure 4 confirms that this is a reasonable value for the slope at $(3, 3)$. So an equation of the tangent to the folium at $(3, 3)$ is

$$y - 3 = -1(x - 3) \quad \text{or} \quad x + y = 6$$

(c) The tangent line is horizontal if $y' = 0$. Using the expression for y' from part (a), we see that $y' = 0$ when $2y - x^2 = 0$ (provided that $y^2 - 2x \neq 0$). Substituting $y = \frac{1}{2}x^2$ in the equation of the curve, we get

$$x^3 + \left(\frac{1}{2}x^2\right)^3 = 6x\left(\frac{1}{2}x^2\right)$$

which simplifies to $x^6 = 16x^3$. Since $x \neq 0$ in the first quadrant, we have $x^3 = 16$. If $x = 16^{1/3} = 2^{4/3}$, then $y = \frac{1}{2}(2^{8/3}) = 2^{5/3}$. Thus the tangent is horizontal at $(2^{4/3}, 2^{5/3})$, which is approximately $(2.5198, 3.1748)$. Looking at Figure 5, we see that our answer is reasonable.

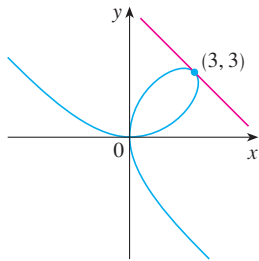


FIGURE 4

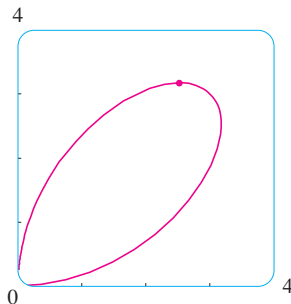


FIGURE 5

NOTE 2 There is a formula for the three roots of a cubic equation that is like the quadratic formula but much more complicated. If we use this formula (or a computer algebra system) to solve the equation $x^3 + y^3 = 6xy$ for y in terms of x , we get three functions determined by the equation:

$$y = f(x) = \sqrt[3]{-\frac{1}{2}x^3 + \sqrt{\frac{1}{4}x^6 - 8x^3}} + \sqrt[3]{-\frac{1}{2}x^3 - \sqrt{\frac{1}{4}x^6 - 8x^3}}$$

and

$$y = \frac{1}{2} \left[-f(x) \pm \sqrt{-3 \left(\sqrt[3]{-\frac{1}{2}x^3 + \sqrt{\frac{1}{4}x^6 - 8x^3}} - \sqrt[3]{-\frac{1}{2}x^3 - \sqrt{\frac{1}{4}x^6 - 8x^3}} \right)} \right]$$

(These are the three functions whose graphs are shown in Figure 3.) You can see that the method of implicit differentiation saves an enormous amount of work in cases such as this. Moreover, implicit differentiation works just as easily for equations such as

$$y^5 + 3x^2y^2 + 5x^4 = 12$$

for which it is *impossible* to find a similar expression for y in terms of x .

EXAMPLE 3 Find y' if $\sin(x + y) = y^2 \cos x$.

SOLUTION Differentiating implicitly with respect to x and remembering that y is a function of x , we get

$$\cos(x + y) \cdot (1 + y') = y^2(-\sin x) + (\cos x)(2yy')$$

(Note that we have used the Chain Rule on the left side and the Product Rule and Chain Rule on the right side.) If we collect the terms that involve y' , we get

$$\cos(x + y) + y^2 \sin x = (2y \cos x)y' - \cos(x + y) \cdot y'$$

So

$$y' = \frac{y^2 \sin x + \cos(x + y)}{2y \cos x - \cos(x + y)}$$

Figure 6, drawn with the implicit-plotting command of a computer algebra system, shows part of the curve $\sin(x + y) = y^2 \cos x$. As a check on our calculation, notice that $y' = -1$ when $x = y = 0$ and it appears from the graph that the slope is approximately -1 at the origin.

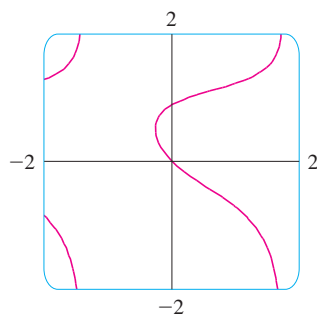


FIGURE 6

Figures 7, 8, and 9 show three more curves produced by a computer algebra system with an implicit-plotting command. In Exercises 41–42 you will have an opportunity to create and examine unusual curves of this nature.

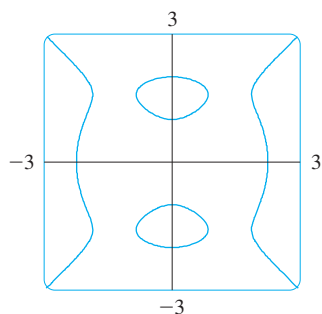


FIGURE 7

$$(y^2 - 1)(y^2 - 4) = x^2(x^2 - 4)$$

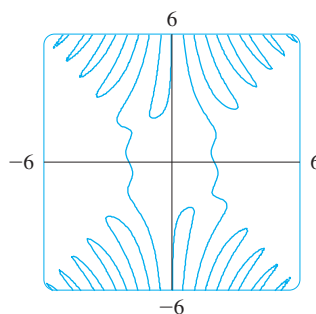


FIGURE 8

$$(y^2 - 1) \sin(xy) = x^2 - 4$$

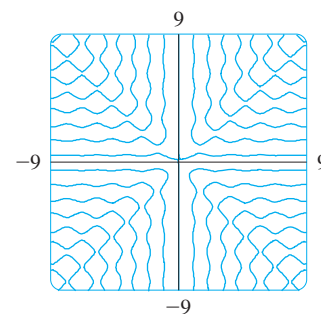


FIGURE 9

$$y \sin 3x = x \cos 3y$$

The following example shows how to find the second derivative of a function that is defined implicitly.

EXAMPLE 4 Find y'' if $x^4 + y^4 = 16$.

SOLUTION Differentiating the equation implicitly with respect to x , we get

$$4x^3 + 4y^3y' = 0$$

Solving for y' gives

$$\boxed{3} \quad y' = -\frac{x^3}{y^3}$$

Figure 10 shows the graph of the curve $x^4 + y^4 = 16$ of Example 4. Notice that it's a stretched and flattened version of the circle $x^2 + y^2 = 4$. For this reason it's sometimes called a *fat circle*. It starts out very steep on the left but quickly becomes very flat. This can be seen from the expression

$$y' = -\frac{x^3}{y^3} = -\left(\frac{x}{y}\right)^3$$

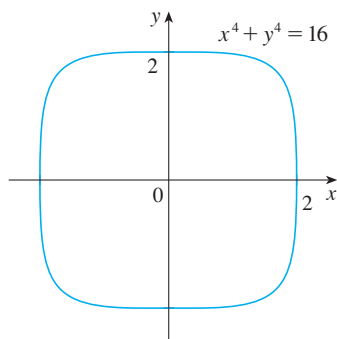


FIGURE 10

To find y'' we differentiate this expression for y' using the Quotient Rule and remembering that y is a function of x :

$$\begin{aligned} y'' &= \frac{d}{dx} \left(-\frac{x^3}{y^3} \right) = -\frac{y^3 (d/dx)(x^3) - x^3 (d/dx)(y^3)}{(y^3)^2} \\ &= -\frac{y^3 \cdot 3x^2 - x^3(3y^2y')}{y^6} \end{aligned}$$

If we now substitute Equation 3 into this expression, we get

$$\begin{aligned} y'' &= -\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6} \\ &= -\frac{3(x^2y^4 + x^6)}{y^7} = -\frac{3x^2(y^4 + x^4)}{y^7} \end{aligned}$$

But the values of x and y must satisfy the original equation $x^4 + y^4 = 16$. So the answer simplifies to

$$y'' = -\frac{3x^2(16)}{y^7} = -48 \frac{x^2}{y^7}$$

Derivatives of Inverse Trigonometric Functions

The inverse trigonometric functions were reviewed in Section 1.6. We discussed their continuity in Section 2.5 and their asymptotes in Section 2.6. Here we use implicit differentiation to find the derivatives of the inverse trigonometric functions, assuming that these functions are differentiable. [In fact, if f is any one-to-one differentiable function, it can be proved that its inverse function f^{-1} is also differentiable, except where its tangents are vertical. This is plausible because the graph of a differentiable function has no corner or kink and so if we reflect it about $y = x$, the graph of its inverse function also has no corner or kink.]

Recall the definition of the arcsine function:

$$y = \sin^{-1}x \quad \text{means} \quad \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

Differentiating $\sin y = x$ implicitly with respect to x , we obtain

$$\cos y \frac{dy}{dx} = 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\cos y}$$

Now $\cos y \geq 0$, since $-\pi/2 \leq y \leq \pi/2$, so

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

The same method can be used to find a formula for the derivative of *any* inverse function. See Exercise 77.

Therefore

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$

Figure 11 shows the graph of $f(x) = \tan^{-1} x$ and its derivative $f'(x) = 1/(1 + x^2)$. Notice that f is increasing and $f'(x)$ is always positive. The fact that $\tan^{-1} x \rightarrow \pm\pi/2$ as $x \rightarrow \pm\infty$ is reflected in the fact that $f'(x) \rightarrow 0$ as $x \rightarrow \pm\infty$.

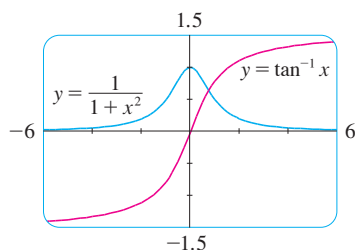


FIGURE 11

The formula for the derivative of the arctangent function is derived in a similar way. If $y = \tan^{-1} x$, then $\tan y = x$. Differentiating this latter equation implicitly with respect to x , we have

$$\sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

$$\frac{d}{dx} (\tan^{-1} x) = \frac{1}{1 + x^2}$$

V EXAMPLE 5 Differentiate (a) $y = \frac{1}{\sin^{-1} x}$ and (b) $f(x) = x \arctan \sqrt{x}$.

SOLUTION

$$\begin{aligned} \text{(a)} \quad \frac{dy}{dx} &= \frac{d}{dx} (\sin^{-1} x)^{-1} = -(\sin^{-1} x)^{-2} \frac{d}{dx} (\sin^{-1} x) \\ &= -\frac{1}{(\sin^{-1} x)^2 \sqrt{1 - x^2}} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad f'(x) &= x \frac{1}{1 + (\sqrt{x})^2} \left(\frac{1}{2} x^{-1/2}\right) + \arctan \sqrt{x} \\ &= \frac{\sqrt{x}}{2(1 + x)} + \arctan \sqrt{x} \end{aligned}$$

Recall that $\arctan x$ is an alternative notation for $\tan^{-1} x$.

The inverse trigonometric functions that occur most frequently are the ones that we have just discussed. The derivatives of the remaining four are given in the following table. The proofs of the formulas are left as exercises.

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx} (\csc^{-1} x) = -\frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx} (\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx} (\sec^{-1} x) = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx} (\tan^{-1} x) = \frac{1}{1 + x^2}$$

$$\frac{d}{dx} (\cot^{-1} x) = -\frac{1}{1 + x^2}$$

The formulas for the derivatives of $\csc^{-1} x$ and $\sec^{-1} x$ depend on the definitions that are used for these functions. See Exercise 64.

3.5 Exercises

1–4

- (a) Find y' by implicit differentiation.
 (b) Solve the equation explicitly for y and differentiate to get y' in terms of x .
 (c) Check that your solutions to parts (a) and (b) are consistent by substituting the expression for y into your solution for part (a).

$$\begin{array}{ll} 1. 9x^2 - y^2 = 1 & 2. 2x^2 + x + xy = 1 \\ 3. \frac{1}{x} + \frac{1}{y} = 1 & 4. \cos x + \sqrt{y} = 5 \end{array}$$

5–20 Find dy/dx by implicit differentiation.

$$\begin{array}{ll} 5. x^3 + y^3 = 1 & 6. 2\sqrt{x} + \sqrt{y} = 3 \\ 7. x^2 + xy - y^2 = 4 & 8. 2x^3 + x^2y - xy^3 = 2 \\ 9. x^4(x + y) = y^2(3x - y) & 10. xe^y = x - y \\ 11. y \cos x = x^2 + y^2 & 12. \cos(xy) = 1 + \sin y \\ 13. 4 \cos x \sin y = 1 & 14. e^y \sin x = x + xy \\ 15. e^{x/y} = x - y & 16. \sqrt{x + y} = 1 + x^2y^2 \\ 17. \tan^{-1}(x^2y) = x + xy^2 & 18. x \sin y + y \sin x = 1 \\ 19. e^y \cos x = 1 + \sin(xy) & 20. \tan(x - y) = \frac{y}{1 + x^2} \end{array}$$

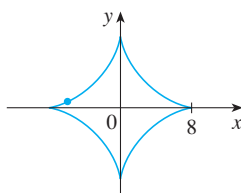
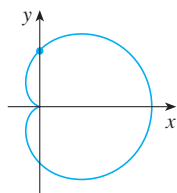
21. If $f(x) + x^2[f(x)]^3 = 10$ and $f(1) = 2$, find $f'(1)$.
 22. If $g(x) + x \sin g(x) = x^2$, find $g'(0)$.

23–24 Regard y as the independent variable and x as the dependent variable and use implicit differentiation to find dx/dy .

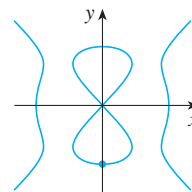
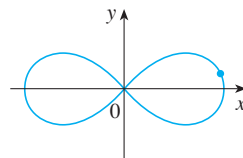
$$23. x^4y^2 - x^3y + 2xy^3 = 0 \quad 24. y \sec x = x \tan x$$

25–32 Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

25. $y \sin 2x = x \cos 2y$, $(\pi/2, \pi/4)$
 26. $\sin(x + y) = 2x - 2y$, (π, π)
 27. $x^2 + xy + y^2 = 3$, $(1, 1)$ (ellipse)
 28. $x^2 + 2xy - y^2 + x = 2$, $(1, 2)$ (hyperbola)
 29. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$, $(0, \frac{1}{2})$ (cardioid)
 30. $x^{2/3} + y^{2/3} = 4$, $(-3\sqrt{3}, 1)$ (astroid)



31. $2(x^2 + y^2)^2 = 25(x^2 - y^2)$, $(3, 1)$ (lemniscate)
 32. $y^2(y^2 - 4) = x^2(x^2 - 5)$, $(0, -2)$ (devil's curve)



33. (a) The curve with equation $y^2 = 5x^4 - x^2$ is called a **kampyle of Eudoxus**. Find an equation of the tangent line to this curve at the point $(1, 2)$.
 (b) Illustrate part (a) by graphing the curve and the tangent line on a common screen. (If your graphing device will graph implicitly defined curves, then use that capability. If not, you can still graph this curve by graphing its upper and lower halves separately.)
 34. (a) The curve with equation $y^2 = x^3 + 3x^2$ is called the **Tschirnhausen cubic**. Find an equation of the tangent line to this curve at the point $(1, -2)$.
 (b) At what points does this curve have horizontal tangents?
 (c) Illustrate parts (a) and (b) by graphing the curve and the tangent lines on a common screen.

35–38 Find y'' by implicit differentiation.

$$\begin{array}{ll} 35. 9x^2 + y^2 = 9 & 36. \sqrt{x} + \sqrt{y} = 1 \\ 37. x^3 + y^3 = 1 & 38. x^4 + y^4 = a^4 \end{array}$$

39. If $xy + e^y = e$, find the value of y'' at the point where $x = 0$.
 40. If $x^2 + xy + y^3 = 1$, find the value of y''' at the point where $x = 1$.

- CAS** 41. Fanciful shapes can be created by using the implicit plotting capabilities of computer algebra systems.
 (a) Graph the curve with equation

$$y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)$$

- At how many points does this curve have horizontal tangents? Estimate the x -coordinates of these points.
 (b) Find equations of the tangent lines at the points $(0, 1)$ and $(0, 2)$.
 (c) Find the exact x -coordinates of the points in part (a).
 (d) Create even more fanciful curves by modifying the equation in part (a).

CAS 42. (a) The curve with equation

$$2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$$

has been likened to a bouncing wagon. Use a computer algebra system to graph this curve and discover why.

- (b) At how many points does this curve have horizontal tangent lines? Find the x -coordinates of these points.

43. Find the points on the lemniscate in Exercise 31 where the tangent is horizontal.

44. Show by implicit differentiation that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

45. Find an equation of the tangent line to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) .

46. Show that the sum of the x - and y -intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c .

47. Show, using implicit differentiation, that any tangent line at a point P to a circle with center O is perpendicular to the radius OP .

48. The Power Rule can be proved using implicit differentiation for the case where n is a rational number, $n = p/q$, and $y = f(x) = x^n$ is assumed beforehand to be a differentiable function. If $y = x^{p/q}$, then $y^q = x^p$. Use implicit differentiation to show that

$$y' = \frac{p}{q} x^{(p/q)-1}$$

49–60 Find the derivative of the function. Simplify where possible.

49. $y = (\tan^{-1}x)^2$

50. $y = \tan^{-1}(x^2)$

51. $y = \sin^{-1}(2x + 1)$

52. $g(x) = \sqrt{x^2 - 1} \sec^{-1}x$

53. $G(x) = \sqrt{1 - x^2} \arccos x$

54. $y = \tan^{-1}(x - \sqrt{1 + x^2})$

55. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t)$

56. $F(\theta) = \arcsin \sqrt{\sin \theta}$

57. $y = x \sin^{-1}x + \sqrt{1 - x^2}$

58. $y = \cos^{-1}(\sin^{-1}t)$

59. $y = \arccos\left(\frac{b + a \cos x}{a + b \cos x}\right), \quad 0 \leq x \leq \pi, \quad a > b > 0$

60. $y = \arctan \sqrt{\frac{1-x}{1+x}}$

61–62 Find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f' .

61. $f(x) = \sqrt{1 - x^2} \arcsin x$

62. $f(x) = \arctan(x^2 - x)$

63. Prove the formula for $(d/dx)(\cos^{-1}x)$ by the same method as for $(d/dx)(\sin^{-1}x)$.

64. (a) One way of defining $\sec^{-1}x$ is to say that $y = \sec^{-1}x \iff \sec y = x$ and $0 \leq y < \pi/2$ or $\pi \leq y < 3\pi/2$. Show that, with this definition,

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}}$$

(b) Another way of defining $\sec^{-1}x$ that is sometimes used is to say that $y = \sec^{-1}x \iff \sec y = x$ and $0 \leq y \leq \pi, y \neq 0$. Show that, with this definition,

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

65–68 Two curves are **orthogonal** if their tangent lines are perpendicular at each point of intersection. Show that the given families of curves are **orthogonal trajectories** of each other; that is, every curve in one family is orthogonal to every curve in the other family. Sketch both families of curves on the same axes.

65. $x^2 + y^2 = r^2, \quad ax + by = 0$

66. $x^2 + y^2 = ax, \quad x^2 + y^2 = by$

67. $y = cx^2, \quad x^2 + 2y^2 = k$

68. $y = ax^3, \quad x^2 + 3y^2 = b$

69. Show that the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the hyperbola $x^2/A^2 - y^2/B^2 = 1$ are orthogonal trajectories if $A^2 < a^2$ and $a^2 - b^2 = A^2 + B^2$ (so the ellipse and hyperbola have the same foci).

70. Find the value of the number a such that the families of curves $y = (x + c)^{-1}$ and $y = a(x + k)^{1/3}$ are orthogonal trajectories.

71. (a) The *van der Waals equation* for n moles of a gas is

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$$

where P is the pressure, V is the volume, and T is the temperature of the gas. The constant R is the universal gas constant and a and b are positive constants that are characteristic of a particular gas. If T remains constant, use implicit differentiation to find dV/dP .

(b) Find the rate of change of volume with respect to pressure of 1 mole of carbon dioxide at a volume of $V = 10$ L and

a pressure of $P = 2.5$ atm. Use $a = 3.592 \text{ L}^2\text{-atm/mole}^2$ and $b = 0.04267 \text{ L/mole}$.

72. (a) Use implicit differentiation to find y' if $x^2 + xy + y^2 + 1 = 0$.
 (b) Plot the curve in part (a). What do you see? Prove that what you see is correct.
 (c) In view of part (b), what can you say about the expression for y' that you found in part (a)?

CAS

73. The equation $x^2 - xy + y^2 = 3$ represents a “rotated ellipse,” that is, an ellipse whose axes are not parallel to the coordinate axes. Find the points at which this ellipse crosses the x -axis and show that the tangent lines at these points are parallel.

74. (a) Where does the normal line to the ellipse $x^2 - xy + y^2 = 3$ at the point $(-1, 1)$ intersect the ellipse a second time?



- (b) Illustrate part (a) by graphing the ellipse and the normal line.

75. Find all points on the curve $x^2y^2 + xy = 2$ where the slope of the tangent line is -1 .

76. Find equations of both the tangent lines to the ellipse $x^2 + 4y^2 = 36$ that pass through the point $(12, 3)$.

77. (a) Suppose f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Use implicit

differentiation to show that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is not 0.

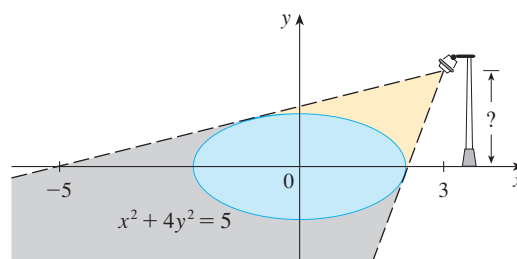
- (b) If $f(4) = 5$ and $f'(4) = \frac{2}{3}$, find $(f^{-1})'(5)$.

78. (a) Show that $f(x) = x + e^x$ is one-to-one.
 (b) What is the value of $f^{-1}(1)$?
 (c) Use the formula from Exercise 77(a) to find $(f^{-1})'(1)$.

79. The **Bessel function** of order 0, $y = J(x)$, satisfies the differential equation $xy'' + y' + xy = 0$ for all values of x and its value at 0 is $J(0) = 1$.

- (a) Find $J'(0)$.
 (b) Use implicit differentiation to find $J''(0)$.

80. The figure shows a lamp located three units to the right of the y -axis and a shadow created by the elliptical region $x^2 + 4y^2 \leq 5$. If the point $(-5, 0)$ is on the edge of the shadow, how far above the x -axis is the lamp located?



LABORATORY PROJECT CAS FAMILIES OF IMPLICIT CURVES

In this project you will explore the changing shapes of implicitly defined curves as you vary the constants in a family, and determine which features are common to all members of the family.

1. Consider the family of curves

$$y^2 - 2x^2(x + 8) = c[(y + 1)^2(y + 9) - x^2]$$

- (a) By graphing the curves with $c = 0$ and $c = 2$, determine how many points of intersection there are. (You might have to zoom in to find all of them.)
 (b) Now add the curves with $c = 5$ and $c = 10$ to your graphs in part (a). What do you notice? What about other values of c ?

2. (a) Graph several members of the family of curves

$$x^2 + y^2 + cx^2y^2 = 1$$

Describe how the graph changes as you change the value of c .

- (b) What happens to the curve when $c = -1$? Describe what appears on the screen. Can you prove it algebraically?
 (c) Find y' by implicit differentiation. For the case $c = -1$, is your expression for y' consistent with what you discovered in part (b)?

CAS

Computer algebra system required

3.6 Derivatives of Logarithmic Functions

In this section we use implicit differentiation to find the derivatives of the logarithmic functions $y = \log_a x$ and, in particular, the natural logarithmic function $y = \ln x$. [It can be proved that logarithmic functions are differentiable; this is certainly plausible from their graphs (see Figure 12 in Section 1.6).]

1

$$\frac{d}{dx} (\log_a x) = \frac{1}{x \ln a}$$

PROOF Let $y = \log_a x$. Then

$$a^y = x$$

Formula 3.4.5 says that

$$\frac{d}{dx} (a^x) = a^x \ln a$$

Differentiating this equation implicitly with respect to x , using Formula 3.4.5, we get

$$a^y (\ln a) \frac{dy}{dx} = 1$$

and so

$$\frac{dy}{dx} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$$

If we put $a = e$ in Formula 1, then the factor $\ln a$ on the right side becomes $\ln e = 1$ and we get the formula for the derivative of the natural logarithmic function $\log_e x = \ln x$:

2

$$\frac{d}{dx} (\ln x) = \frac{1}{x}$$

By comparing Formulas 1 and 2, we see one of the main reasons that natural logarithms (logarithms with base e) are used in calculus: The differentiation formula is simplest when $a = e$ because $\ln e = 1$.

V EXAMPLE 1 Differentiate $y = \ln(x^3 + 1)$.

SOLUTION To use the Chain Rule, we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = \frac{1}{u} \frac{du}{dx} \\ &= \frac{1}{x^3 + 1} (3x^2) = \frac{3x^2}{x^3 + 1} \end{aligned}$$

In general, if we combine Formula 2 with the Chain Rule as in Example 1, we get

3

$$\frac{d}{dx} (\ln u) = \frac{1}{u} \frac{du}{dx}$$

or

$$\frac{d}{dx} [\ln g(x)] = \frac{g'(x)}{g(x)}$$

EXAMPLE 2 Find $\frac{d}{dx} \ln(\sin x)$.

SOLUTION Using [3], we have

$$\frac{d}{dx} \ln(\sin x) = \frac{1}{\sin x} \frac{d}{dx} (\sin x) = \frac{1}{\sin x} \cos x = \cot x$$

EXAMPLE 3 Differentiate $f(x) = \sqrt{\ln x}$.

SOLUTION This time the logarithm is the inner function, so the Chain Rule gives

$$f'(x) = \frac{1}{2}(\ln x)^{-1/2} \frac{d}{dx} (\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$

EXAMPLE 4 Differentiate $f(x) = \log_{10}(2 + \sin x)$.

SOLUTION Using Formula 1 with $a = 10$, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx} \log_{10}(2 + \sin x) \\ &= \frac{1}{(2 + \sin x) \ln 10} \frac{d}{dx} (2 + \sin x) \\ &= \frac{\cos x}{(2 + \sin x) \ln 10} \end{aligned}$$

EXAMPLE 5 Find $\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}}$.

SOLUTION 1

$$\begin{aligned} \frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}} &= \frac{1}{\frac{x+1}{\sqrt{x-2}}} \frac{d}{dx} \frac{x+1}{\sqrt{x-2}} \\ &= \frac{\sqrt{x-2}}{x+1} \frac{\sqrt{x-2} \cdot 1 - (x+1)\left(\frac{1}{2}\right)(x-2)^{-1/2}}{x-2} \\ &= \frac{x-2 - \frac{1}{2}(x+1)}{(x+1)(x-2)} \\ &= \frac{x-5}{2(x+1)(x-2)} \end{aligned}$$

Figure 1 shows the graph of the function f of Example 5 together with the graph of its derivative. It gives a visual check on our calculation. Notice that $f'(x)$ is large negative when f is rapidly decreasing.

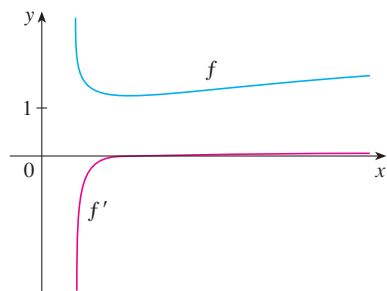


FIGURE 1

SOLUTION 2 If we first simplify the given function using the laws of logarithms, then the differentiation becomes easier:

$$\begin{aligned} \frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}} &= \frac{d}{dx} \left[\ln(x+1) - \frac{1}{2} \ln(x-2) \right] \\ &= \frac{1}{x+1} - \frac{1}{2} \left(\frac{1}{x-2} \right) \end{aligned}$$

(This answer can be left as written, but if we used a common denominator we would see that it gives the same answer as in Solution 1.)

Figure 2 shows the graph of the function $f(x) = \ln|x|$ in Example 6 and its derivative $f'(x) = 1/x$. Notice that when x is small, the graph of $y = \ln|x|$ is steep and so $f'(x)$ is large (positive or negative).

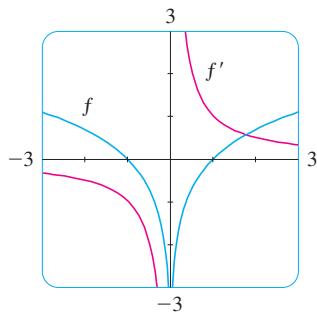


FIGURE 2

V EXAMPLE 6 Find $f'(x)$ if $f(x) = \ln|x|$.

SOLUTION Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

it follows that

$$f'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{-x}(-1) = \frac{1}{x} & \text{if } x < 0 \end{cases}$$

Thus $f'(x) = 1/x$ for all $x \neq 0$.

The result of Example 6 is worth remembering:

4

$$\frac{d}{dx} \ln|x| = \frac{1}{x}$$

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation**.

EXAMPLE 7 Differentiate $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

SOLUTION We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \frac{3}{4} \ln x + \frac{1}{2} \ln(x^2 + 1) - 5 \ln(3x + 2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y} \frac{dy}{dx} = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2 + 1} - 5 \cdot \frac{3}{3x + 2}$$

Solving for dy/dx , we get

$$\frac{dy}{dx} = y \left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2} \right)$$

If we hadn't used logarithmic differentiation in Example 7, we would have had to use both the Quotient Rule and the Product Rule. The resulting calculation would have been horrendous.

Because we have an explicit expression for y , we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5} \left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2} \right)$$

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.
2. Differentiate implicitly with respect to x .
3. Solve the resulting equation for y' .

If $f(x) < 0$ for some values of x , then $\ln f(x)$ is not defined, but we can write $|y| = |f(x)|$ and use Equation 4. We illustrate this procedure by proving the general version of the Power Rule, as promised in Section 3.1.

The Power Rule If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

PROOF Let $y = x^n$ and use logarithmic differentiation:

$$\ln |y| = \ln |x|^n = n \ln |x| \quad x \neq 0$$


Therefore

$$\frac{y'}{y} = \frac{n}{x}$$

Hence

$$y' = n \frac{y}{x} = n \frac{x^n}{x} = nx^{n-1}$$

If $x = 0$, we can show that $f'(0) = 0$ for $n > 1$ directly from the definition of a derivative.

 You should distinguish carefully between the Power Rule $[(x^n)' = nx^{n-1}]$, where the base is variable and the exponent is constant, and the rule for differentiating exponential functions $[(a^x)' = a^x \ln a]$, where the base is constant and the exponent is variable.

In general there are four cases for exponents and bases:

Constant base, constant exponent

$$1. \frac{d}{dx}(a^b) = 0 \quad (a \text{ and } b \text{ are constants})$$

Variable base, constant exponent

$$2. \frac{d}{dx}[f(x)]^b = b[f(x)]^{b-1}f'(x)$$

Constant base, variable exponent

$$3. \frac{d}{dx}[a^{g(x)}] = a^{g(x)}(\ln a)g'(x)$$

Variable base, variable exponent

$$4. \text{ To find } (d/dx)[f(x)]^{g(x)}, \text{ logarithmic differentiation can be used, as in the next example.}$$

V EXAMPLE 8 Differentiate $y = x^{\sqrt{x}}$.

SOLUTION 1 Since both the base and the exponent are variable, we use logarithmic differentiation:

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x} \ln x$$

$$\frac{y'}{y} = \sqrt{x} \cdot \frac{1}{x} + (\ln x) \frac{1}{2\sqrt{x}}$$

$$y' = y \left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}} \right) = x^{\sqrt{x}} \left(\frac{2 + \ln x}{2\sqrt{x}} \right)$$

Figure 3 illustrates Example 8 by showing the graphs of $f(x) = x^{\sqrt{x}}$ and its derivative.

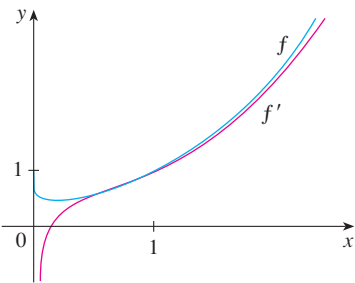


FIGURE 3

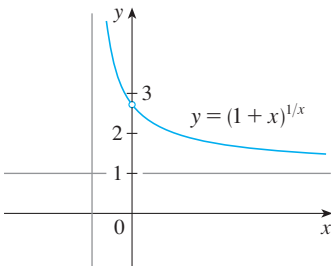


FIGURE 4

x	$(1 + x)^{1/x}$
0.1	2.59374246
0.01	2.70481383
0.001	2.71692393
0.0001	2.71814593
0.00001	2.71826824
0.000001	2.71828047
0.0000001	2.71828169
0.00000001	2.71828181

SOLUTION 2 Another method is to write $x^{\sqrt{x}} = (e^{\ln x})^{\sqrt{x}}$:

$$\begin{aligned}\frac{d}{dx}(x^{\sqrt{x}}) &= \frac{d}{dx}(e^{\sqrt{x} \ln x}) = e^{\sqrt{x} \ln x} \frac{d}{dx}(\sqrt{x} \ln x) \\ &= x^{\sqrt{x}} \left(\frac{2 + \ln x}{2\sqrt{x}} \right) \quad (\text{as in Solution 1})\end{aligned}$$

The Number e as a Limit

We have shown that if $f(x) = \ln x$, then $f'(x) = 1/x$. Thus $f'(1) = 1$. We now use this fact to express the number e as a limit.

From the definition of a derivative as a limit, we have

$$\begin{aligned}f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \rightarrow 0} \frac{f(1+x) - f(1)}{x} \\ &= \lim_{x \rightarrow 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \rightarrow 0} \frac{1}{x} \ln(1+x) \\ &= \lim_{x \rightarrow 0} \ln(1+x)^{1/x}\end{aligned}$$

Because $f'(1) = 1$, we have

$$\lim_{x \rightarrow 0} \ln(1+x)^{1/x} = 1$$

Then, by Theorem 2.5.8 and the continuity of the exponential function, we have

$$e = e^1 = e^{\lim_{x \rightarrow 0} \ln(1+x)^{1/x}} = \lim_{x \rightarrow 0} e^{\ln(1+x)^{1/x}} = \lim_{x \rightarrow 0} (1+x)^{1/x}$$

5

$$e = \lim_{x \rightarrow 0} (1+x)^{1/x}$$

Formula 5 is illustrated by the graph of the function $y = (1+x)^{1/x}$ in Figure 4 and a table of values for small values of x . This illustrates the fact that, correct to seven decimal places,

$$e \approx 2.7182818$$

If we put $n = 1/x$ in Formula 5, then $n \rightarrow \infty$ as $x \rightarrow 0^+$ and so an alternative expression for e is

6

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n$$

3.6 Exercises

1. Explain why the natural logarithmic function $y = \ln x$ is used much more frequently in calculus than the other logarithmic functions $y = \log_a x$.

2–22 Differentiate the function.

- | | |
|--|---|
| 2. $f(x) = x \ln x - x$ | 4. $f(x) = \ln(\sin^2 x)$ |
| 3. $f(x) = \sin(\ln x)$ | 6. $y = \frac{1}{\ln x}$ |
| 5. $f(x) = \ln \frac{1}{x}$ | 8. $f(x) = \log_5(xe^x)$ |
| 7. $f(x) = \log_{10}(x^3 + 1)$ | 10. $f(u) = \frac{u}{1 + \ln u}$ |
| 9. $f(x) = \sin x \ln(5x)$ | 12. $h(x) = \ln(x + \sqrt{x^2 - 1})$ |
| 11. $g(x) = \ln(x\sqrt{x^2 - 1})$ | 14. $g(r) = r^2 \ln(2r + 1)$ |
| 13. $G(y) = \ln \frac{(2y + 1)^5}{\sqrt{y^2 + 1}}$ | 16. $y = \ln 1 + t - t^3 $ |
| 15. $F(s) = \ln \ln s$ | 18. $y = \ln \cos(\ln x) $ |
| 17. $y = \tan[\ln(ax + b)]$ | 20. $H(z) = \ln \sqrt{\frac{a^2 - z^2}{a^2 + z^2}}$ |
| 19. $y = \ln(e^{-x} + xe^{-x})$ | 22. $y = \log_2(e^{-x} \cos \pi x)$ |
| 21. $y = 2x \log_{10} \sqrt{x}$ | |

23–26 Find y' and y'' .

- | | |
|-----------------------------------|--------------------------------|
| 23. $y = x^2 \ln(2x)$ | 24. $y = \frac{\ln x}{x^2}$ |
| 25. $y = \ln(x + \sqrt{1 + x^2})$ | 26. $y = \ln(\sec x + \tan x)$ |

27–30 Differentiate f and find the domain of f .

- | | |
|---------------------------------------|-------------------------------|
| 27. $f(x) = \frac{x}{1 - \ln(x - 1)}$ | 28. $f(x) = \sqrt{2 + \ln x}$ |
| 29. $f(x) = \ln(x^2 - 2x)$ | 30. $f(x) = \ln \ln x$ |

31. If $f(x) = \frac{\ln x}{x^2}$, find $f'(1)$.

32. If $f(x) = \ln(1 + e^{2x})$, find $f'(0)$.

33–34 Find an equation of the tangent line to the curve at the given point.

33. $y = \ln(x^2 - 3x + 1)$, $(3, 0)$ 34. $y = x^2 \ln x$, $(1, 0)$

35. If $f(x) = \sin x + \ln x$, find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f' .
36. Find equations of the tangent lines to the curve $y = (\ln x)/x$ at the points $(1, 0)$ and $(e, 1/e)$. Illustrate by graphing the curve and its tangent lines.

37. Let $f(x) = cx + \ln(\cos x)$. For what value of c is $f'(\pi/4) = 6$?

38. Let $f(x) = \log_a(3x^2 - 2)$. For what value of a is $f'(1) = 3$?

39–50 Use logarithmic differentiation to find the derivative of the function.

- | | |
|------------------------------------|---|
| 39. $y = (x^2 + 2)^2(x^4 + 4)^4$ | 40. $y = \frac{e^{-x} \cos^2 x}{x^2 + x + 1}$ |
| 41. $y = \sqrt{\frac{x-1}{x^4+1}}$ | 42. $y = \sqrt{x} e^{x^2-x} (x+1)^{2/3}$ |
| 43. $y = x^x$ | 44. $y = x^{\cos x}$ |
| 45. $y = x^{\sin x}$ | 46. $y = \sqrt{x}^x$ |
| 47. $y = (\cos x)^x$ | 48. $y = (\sin x)^{\ln x}$ |
| 49. $y = (\tan x)^{1/x}$ | 50. $y = (\ln x)^{\cos x}$ |

51. Find y' if $y = \ln(x^2 + y^2)$.

52. Find y' if $x^y = y^x$.

53. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

54. Find $\frac{d^9}{dx^9}(x^8 \ln x)$.

55. Use the definition of derivative to prove that

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$$

56. Show that $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

3.7 Rates of Change in the Natural and Social Sciences

We know that if $y = f(x)$, then the derivative dy/dx can be interpreted as the rate of change of y with respect to x . In this section we examine some of the applications of this idea to physics, chemistry, biology, economics, and other sciences.

Let's recall from Section 2.7 the basic idea behind rates of change. If x changes from x_1 to x_2 , then the change in x is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in Figure 1. Its limit as $\Delta x \rightarrow 0$ is the derivative $f'(x_1)$, which can therefore be interpreted as the **instantaneous rate of change of y with respect to x** or the slope of the tangent line at $P(x_1, f(x_1))$. Using Leibniz notation, we write the process in the form

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

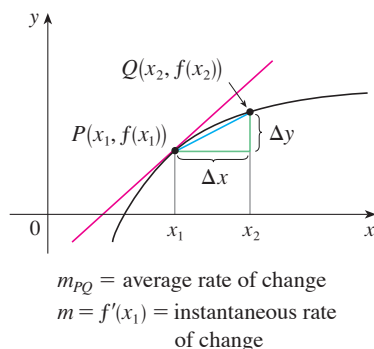


FIGURE 1

Whenever the function $y = f(x)$ has a specific interpretation in one of the sciences, its derivative will have a specific interpretation as a rate of change. (As we discussed in Section 2.7, the units for dy/dx are the units for y divided by the units for x .) We now look at some of these interpretations in the natural and social sciences.

Physics

If $s = f(t)$ is the position function of a particle that is moving in a straight line, then $\Delta s / \Delta t$ represents the average velocity over a time period Δt , and $v = ds/dt$ represents the instantaneous **velocity** (the rate of change of displacement with respect to time). The instantaneous rate of change of velocity with respect to time is **acceleration**: $a(t) = v'(t) = s''(t)$. This was discussed in Sections 2.7 and 2.8, but now that we know the differentiation formulas, we are able to solve problems involving the motion of objects more easily.

V EXAMPLE 1 The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in meters.

- Find the velocity at time t .
- What is the velocity after 2 s? After 4 s?
- When is the particle at rest?
- When is the particle moving forward (that is, in the positive direction)?
- Draw a diagram to represent the motion of the particle.
- Find the total distance traveled by the particle during the first five seconds.
- Find the acceleration at time t and after 4 s.

- (h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 5$.
 (i) When is the particle speeding up? When is it slowing down?

SOLUTION

- (a) The velocity function is the derivative of the position function.

$$s = f(t) = t^3 - 6t^2 + 9t$$

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$$

- (b) The velocity after 2 s means the instantaneous velocity when $t = 2$, that is,

$$v(2) = \left. \frac{ds}{dt} \right|_{t=2} = 3(2)^2 - 12(2) + 9 = -3 \text{ m/s}$$

The velocity after 4 s is

$$v(4) = 3(4)^2 - 12(4) + 9 = 9 \text{ m/s}$$

- (c) The particle is at rest when $v(t) = 0$, that is,

$$3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t - 1)(t - 3) = 0$$

and this is true when $t = 1$ or $t = 3$. Thus the particle is at rest after 1 s and after 3 s.

- (d) The particle moves in the positive direction when $v(t) > 0$, that is,

$$3t^2 - 12t + 9 = 3(t - 1)(t - 3) > 0$$

This inequality is true when both factors are positive ($t > 3$) or when both factors are negative ($t < 1$). Thus the particle moves in the positive direction in the time intervals $t < 1$ and $t > 3$. It moves backward (in the negative direction) when $1 < t < 3$.

- (e) Using the information from part (d) we make a schematic sketch in Figure 2 of the motion of the particle back and forth along a line (the s -axis).

- (f) Because of what we learned in parts (d) and (e), we need to calculate the distances traveled during the time intervals $[0, 1]$, $[1, 3]$, and $[3, 5]$ separately.

The distance traveled in the first second is

$$|f(1) - f(0)| = |4 - 0| = 4 \text{ m}$$

From $t = 1$ to $t = 3$ the distance traveled is

$$|f(3) - f(1)| = |0 - 4| = 4 \text{ m}$$

From $t = 3$ to $t = 5$ the distance traveled is

$$|f(5) - f(3)| = |20 - 0| = 20 \text{ m}$$

The total distance is $4 + 4 + 20 = 28 \text{ m}$.

- (g) The acceleration is the derivative of the velocity function:

$$a(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt} = 6t - 12$$

$$a(4) = 6(4) - 12 = 12 \text{ m/s}^2$$

- (h) Figure 3 shows the graphs of s , v , and a .

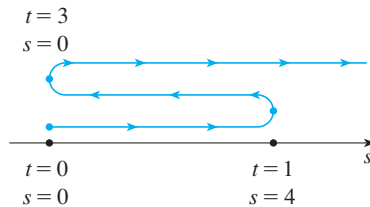


FIGURE 2

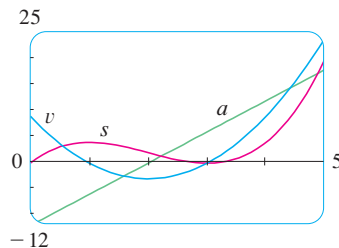


FIGURE 3

(i) The particle speeds up when the velocity is positive and increasing (v and a are both positive) and also when the velocity is negative and decreasing (v and a are both negative). In other words, the particle speeds up when the velocity and acceleration have the same sign. (The particle is pushed in the same direction it is moving.) From Figure 3 we see that this happens when $1 < t < 2$ and when $t > 3$. The particle slows down when v and a have opposite signs, that is, when $0 \leq t < 1$ and when $2 < t < 3$. Figure 4 summarizes the motion of the particle.

TEC In Module 3.7 you can see an animation of Figure 4 with an expression for s that you can choose yourself.

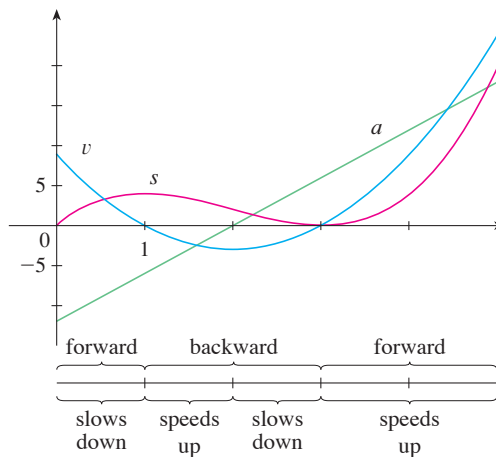


FIGURE 4

EXAMPLE 2 If a rod or piece of wire is homogeneous, then its linear density is uniform and is defined as the mass per unit length ($\rho = m/l$) and measured in kilograms per meter. Suppose, however, that the rod is not homogeneous but that its mass measured from its left end to a point x is $m = f(x)$, as shown in Figure 5.

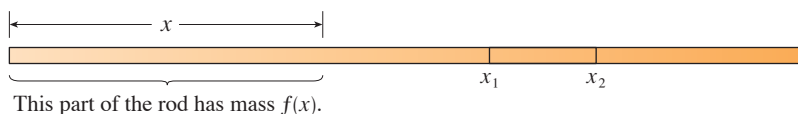


FIGURE 5

The mass of the part of the rod that lies between $x = x_1$ and $x = x_2$ is given by $\Delta m = f(x_2) - f(x_1)$, so the average density of that part of the rod is

$$\text{average density} = \frac{\Delta m}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

If we now let $\Delta x \rightarrow 0$ (that is, $x_2 \rightarrow x_1$), we are computing the average density over smaller and smaller intervals. The **linear density** ρ at x_1 is the limit of these average densities as $\Delta x \rightarrow 0$; that is, the linear density is the rate of change of mass with respect to length. Symbolically,

$$\rho = \lim_{\Delta x \rightarrow 0} \frac{\Delta m}{\Delta x} = \frac{dm}{dx}$$

Thus the linear density of the rod is the derivative of mass with respect to length.

For instance, if $m = f(x) = \sqrt{x}$, where x is measured in meters and m in kilograms, then the average density of the part of the rod given by $1 \leq x \leq 1.2$ is

$$\frac{\Delta m}{\Delta x} = \frac{f(1.2) - f(1)}{1.2 - 1} = \frac{\sqrt{1.2} - 1}{0.2} \approx 0.48 \text{ kg/m}$$

while the density right at $x = 1$ is

$$\rho = \left. \frac{dm}{dx} \right|_{x=1} = \left. \frac{1}{2\sqrt{x}} \right|_{x=1} = 0.50 \text{ kg/m}$$

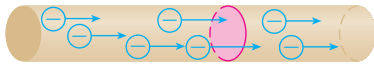


FIGURE 6

V EXAMPLE 3 A current exists whenever electric charges move. Figure 6 shows part of a wire and electrons moving through a plane surface, shaded red. If ΔQ is the net charge that passes through this surface during a time period Δt , then the average current during this time interval is defined as

$$\text{average current} = \frac{\Delta Q}{\Delta t} = \frac{Q_2 - Q_1}{t_2 - t_1}$$

If we take the limit of this average current over smaller and smaller time intervals, we get what is called the **current** I at a given time t_1 :

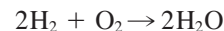
$$I = \lim_{\Delta t \rightarrow 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

Thus the current is the rate at which charge flows through a surface. It is measured in units of charge per unit time (often coulombs per second, called amperes).

Velocity, density, and current are not the only rates of change that are important in physics. Others include power (the rate at which work is done), the rate of heat flow, temperature gradient (the rate of change of temperature with respect to position), and the rate of decay of a radioactive substance in nuclear physics.

Chemistry

EXAMPLE 4 A chemical reaction results in the formation of one or more substances (called *products*) from one or more starting materials (called *reactants*). For instance, the “equation”



indicates that two molecules of hydrogen and one molecule of oxygen form two molecules of water. Let’s consider the reaction



where A and B are the reactants and C is the product. The **concentration** of a reactant A is the number of moles (1 mole = 6.022×10^{23} molecules) per liter and is denoted by $[A]$. The concentration varies during a reaction, so $[A]$, $[B]$, and $[C]$ are all functions of

time (t). The average rate of reaction of the product C over a time interval $t_1 \leq t \leq t_2$ is

$$\frac{\Delta[C]}{\Delta t} = \frac{[C](t_2) - [C](t_1)}{t_2 - t_1}$$

But chemists are more interested in the **instantaneous rate of reaction**, which is obtained by taking the limit of the average rate of reaction as the time interval Δt approaches 0:

$$\text{rate of reaction} = \lim_{\Delta t \rightarrow 0} \frac{\Delta[C]}{\Delta t} = \frac{d[C]}{dt}$$

Since the concentration of the product increases as the reaction proceeds, the derivative $d[C]/dt$ will be positive, and so the rate of reaction of C is positive. The concentrations of the reactants, however, decrease during the reaction, so, to make the rates of reaction of A and B positive numbers, we put minus signs in front of the derivatives $d[A]/dt$ and $d[B]/dt$. Since [A] and [B] each decrease at the same rate that [C] increases, we have

$$\text{rate of reaction} = \frac{d[C]}{dt} = -\frac{d[A]}{dt} = -\frac{d[B]}{dt}$$

More generally, it turns out that for a reaction of the form



we have

$$-\frac{1}{a} \frac{d[A]}{dt} = -\frac{1}{b} \frac{d[B]}{dt} = \frac{1}{c} \frac{d[C]}{dt} = \frac{1}{d} \frac{d[D]}{dt}$$

The rate of reaction can be determined from data and graphical methods. In some cases there are explicit formulas for the concentrations as functions of time, which enable us to compute the rate of reaction (see Exercise 24).

EXAMPLE 5 One of the quantities of interest in thermodynamics is compressibility. If a given substance is kept at a constant temperature, then its volume V depends on its pressure P . We can consider the rate of change of volume with respect to pressure—namely, the derivative dV/dP . As P increases, V decreases, so $dV/dP < 0$. The **compressibility** is defined by introducing a minus sign and dividing this derivative by the volume V :

$$\text{isothermal compressibility} = \beta = -\frac{1}{V} \frac{dV}{dP}$$

Thus β measures how fast, per unit volume, the volume of a substance decreases as the pressure on it increases at constant temperature.

For instance, the volume V (in cubic meters) of a sample of air at 25°C was found to be related to the pressure P (in kilopascals) by the equation

$$V = \frac{5.3}{P}$$

The rate of change of V with respect to P when $P = 50$ kPa is

$$\begin{aligned}\left.\frac{dV}{dP}\right|_{P=50} &= -\left.\frac{5.3}{P^2}\right|_{P=50} \\ &= -\frac{5.3}{2500} = -0.00212 \text{ m}^3/\text{kPa}\end{aligned}$$

The compressibility at that pressure is

$$\beta = -\frac{1}{V} \left.\frac{dV}{dP}\right|_{P=50} = \frac{0.00212}{\frac{5.3}{50}} = 0.02 \text{ (m}^3/\text{kPa)/m}^3$$

Biology

EXAMPLE 6 Let $n = f(t)$ be the number of individuals in an animal or plant population at time t . The change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$, and so the average rate of growth during the time period $t_1 \leq t \leq t_2$ is

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

The **instantaneous rate of growth** is obtained from this average rate of growth by letting the time period Δt approach 0:

$$\text{growth rate} = \lim_{\Delta t \rightarrow 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

Strictly speaking, this is not quite accurate because the actual graph of a population function $n = f(t)$ would be a step function that is discontinuous whenever a birth or death occurs and therefore not differentiable. However, for a large animal or plant population, we can replace the graph by a smooth approximating curve as in Figure 7.

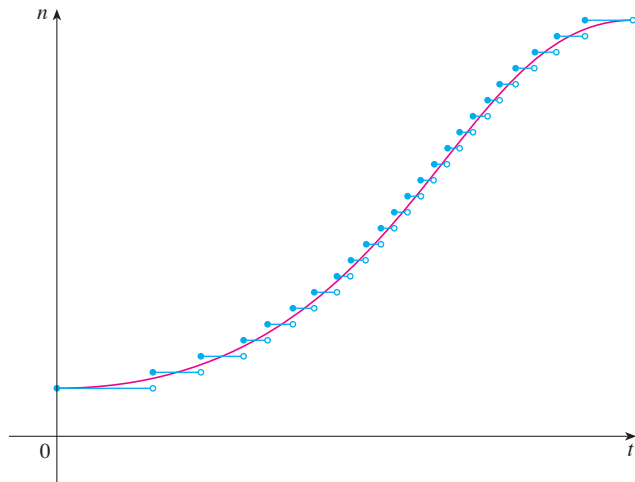
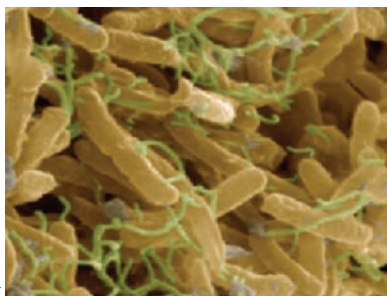


FIGURE 7
A smooth curve approximating
a growth function



E. coli bacteria are about 2 micrometers (μm) long and $0.75 \mu\text{m}$ wide. The image was produced with a scanning electron microscope.

To be more specific, consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour. If the initial population is n_0 and the time t is measured in hours, then

$$f(1) = 2f(0) = 2n_0$$

$$f(2) = 2f(1) = 2^2n_0$$

$$f(3) = 2f(2) = 2^3n_0$$

and, in general,

$$f(t) = 2^t n_0$$

The population function is $n = n_0 2^t$.

In Section 3.4 we showed that

$$\frac{d}{dx}(a^x) = a^x \ln a$$

So the rate of growth of the bacteria population at time t is

$$\frac{dn}{dt} = \frac{d}{dt}(n_0 2^t) = n_0 2^t \ln 2$$

For example, suppose that we start with an initial population of $n_0 = 100$ bacteria. Then the rate of growth after 4 hours is

$$\left. \frac{dn}{dt} \right|_{t=4} = 100 \cdot 2^4 \ln 2 = 1600 \ln 2 \approx 1109$$

This means that, after 4 hours, the bacteria population is growing at a rate of about 1109 bacteria per hour.

EXAMPLE 7 When we consider the flow of blood through a blood vessel, such as a vein or artery, we can model the shape of the blood vessel by a cylindrical tube with radius R and length l as illustrated in Figure 8.

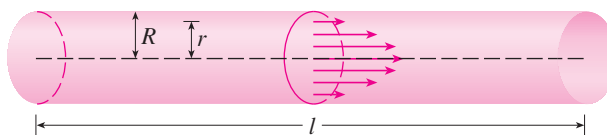


FIGURE 8

Blood flow in an artery

Because of friction at the walls of the tube, the velocity v of the blood is greatest along the central axis of the tube and decreases as the distance r from the axis increases until v becomes 0 at the wall. The relationship between v and r is given by the **law of laminar flow** discovered by the French physician Jean-Louis-Marie Poiseuille in 1840. This law states that

1

$$v = \frac{P}{4\eta l} (R^2 - r^2)$$

where η is the viscosity of the blood and P is the pressure difference between the ends of the tube. If P and l are constant, then v is a function of r with domain $[0, R]$.

For more detailed information, see W. Nichols and M. O'Rourke (eds.), *McDonald's Blood Flow in Arteries: Theoretical, Experimental, and Clinical Principles*, 5th ed. (New York, 2005).

The average rate of change of the velocity as we move from $r = r_1$ outward to $r = r_2$ is given by

$$\frac{\Delta v}{\Delta r} = \frac{v(r_2) - v(r_1)}{r_2 - r_1}$$

and if we let $\Delta r \rightarrow 0$, we obtain the **velocity gradient**, that is, the instantaneous rate of change of velocity with respect to r :

$$\text{velocity gradient} = \lim_{\Delta r \rightarrow 0} \frac{\Delta v}{\Delta r} = \frac{dv}{dr}$$

Using Equation 1, we obtain

$$\frac{dv}{dr} = \frac{P}{4\eta l} (0 - 2r) = -\frac{Pr}{2\eta l}$$

For one of the smaller human arteries we can take $\eta = 0.027$, $R = 0.008$ cm, $l = 2$ cm, and $P = 4000$ dynes/cm², which gives

$$\begin{aligned} v &= \frac{4000}{4(0.027)^2} (0.000064 - r^2) \\ &\approx 1.85 \times 10^4 (6.4 \times 10^{-5} - r^2) \end{aligned}$$

At $r = 0.002$ cm the blood is flowing at a speed of

$$\begin{aligned} v(0.002) &\approx 1.85 \times 10^4 (64 \times 10^{-6} - 4 \times 10^{-6}) \\ &= 1.11 \text{ cm/s} \end{aligned}$$

and the velocity gradient at that point is

$$\left. \frac{dv}{dr} \right|_{r=0.002} = -\frac{4000(0.002)}{2(0.027)^2} \approx -74 \text{ (cm/s)/cm}$$

To get a feeling for what this statement means, let's change our units from centimeters to micrometers (1 cm = 10,000 μm). Then the radius of the artery is 80 μm . The velocity at the central axis is 11,850 $\mu\text{m/s}$, which decreases to 11,110 $\mu\text{m/s}$ at a distance of $r = 20$ μm . The fact that $dv/dr = -74$ ($\mu\text{m/s}$)/ μm means that, when $r = 20$ μm , the velocity is decreasing at a rate of about 74 $\mu\text{m/s}$ for each micrometer that we proceed away from the center.

Economics

V EXAMPLE 8 Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity. The function C is called a **cost function**. If the number of items produced is increased from x_1 to x_2 , then the additional cost is $\Delta C = C(x_2) - C(x_1)$, and the average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

The limit of this quantity as $\Delta x \rightarrow 0$, that is, the instantaneous rate of change of cost

with respect to the number of items produced, is called the **marginal cost** by economists:

$$\text{marginal cost} = \lim_{\Delta x \rightarrow 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

[Since x often takes on only integer values, it may not make literal sense to let Δx approach 0, but we can always replace $C(x)$ by a smooth approximating function as in Example 6.]

Taking $\Delta x = 1$ and n large (so that Δx is small compared to n), we have

$$C'(n) \approx C(n + 1) - C(n)$$

Thus the marginal cost of producing n units is approximately equal to the cost of producing one more unit [the $(n + 1)$ st unit].

It is often appropriate to represent a total cost function by a polynomial

$$C(x) = a + bx + cx^2 + dx^3$$

where a represents the overhead cost (rent, heat, maintenance) and the other terms represent the cost of raw materials, labor, and so on. (The cost of raw materials may be proportional to x , but labor costs might depend partly on higher powers of x because of overtime costs and inefficiencies involved in large-scale operations.)

For instance, suppose a company has estimated that the cost (in dollars) of producing x items is

$$C(x) = 10,000 + 5x + 0.01x^2$$

Then the marginal cost function is

$$C'(x) = 5 + 0.02x$$

The marginal cost at the production level of 500 items is

$$C'(500) = 5 + 0.02(500) = \$15/\text{item}$$

This gives the rate at which costs are increasing with respect to the production level when $x = 500$ and predicts the cost of the 501st item.

The actual cost of producing the 501st item is

$$\begin{aligned} C(501) - C(500) &= [10,000 + 5(501) + 0.01(501)^2] \\ &\quad - [10,000 + 5(500) + 0.01(500)^2] \\ &= \$15.01 \end{aligned}$$

Notice that $C'(500) \approx C(501) - C(500)$.

Economists also study marginal demand, marginal revenue, and marginal profit, which are the derivatives of the demand, revenue, and profit functions. These will be considered in Chapter 4 after we have developed techniques for finding the maximum and minimum values of functions.

Other Sciences

Rates of change occur in all the sciences. A geologist is interested in knowing the rate at which an intruded body of molten rock cools by conduction of heat into surrounding rocks. An engineer wants to know the rate at which water flows into or out of a reservoir. An

urban geographer is interested in the rate of change of the population density in a city as the distance from the city center increases. A meteorologist is concerned with the rate of change of atmospheric pressure with respect to height (see Exercise 17 in Section 3.8).

In psychology, those interested in learning theory study the so-called learning curve, which graphs the performance $P(t)$ of someone learning a skill as a function of the training time t . Of particular interest is the rate at which performance improves as time passes, that is, dP/dt .

In sociology, differential calculus is used in analyzing the spread of rumors (or innovations or fads or fashions). If $p(t)$ denotes the proportion of a population that knows a rumor by time t , then the derivative dp/dt represents the rate of spread of the rumor (see Exercise 84 in Section 3.4).


A Single Idea, Many Interpretations

Velocity, density, current, power, and temperature gradient in physics; rate of reaction and compressibility in chemistry; rate of growth and blood velocity gradient in biology; marginal cost and marginal profit in economics; rate of heat flow in geology; rate of improvement of performance in psychology; rate of spread of a rumor in sociology—these are all special cases of a single mathematical concept, the derivative.

This is an illustration of the fact that part of the power of mathematics lies in its abstractness. A single abstract mathematical concept (such as the derivative) can have different interpretations in each of the sciences. When we develop the properties of the mathematical concept once and for all, we can then turn around and apply these results to all of the sciences. This is much more efficient than developing properties of special concepts in each separate science. The French mathematician Joseph Fourier (1768–1830) put it succinctly: “Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them.”

3.7 Exercises

1–4 A particle moves according to a law of motion $s = f(t)$, $t \geq 0$, where t is measured in seconds and s in feet.

- Find the velocity at time t .
- What is the velocity after 3 s?
- When is the particle at rest?
- When is the particle moving in the positive direction?
- Find the total distance traveled during the first 8 s.
- Draw a diagram like Figure 2 to illustrate the motion of the particle.
- Find the acceleration at time t and after 3 s.
-  Graph the position, velocity, and acceleration functions for $0 \leq t \leq 8$.
- When is the particle speeding up? When is it slowing down?

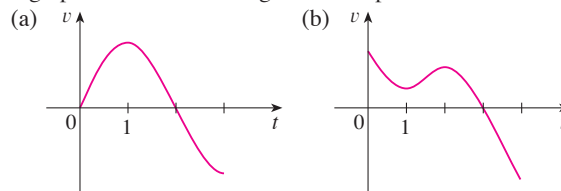
1. $f(t) = t^3 - 12t^2 + 36t$

2. $f(t) = 0.01t^4 - 0.04t^3$

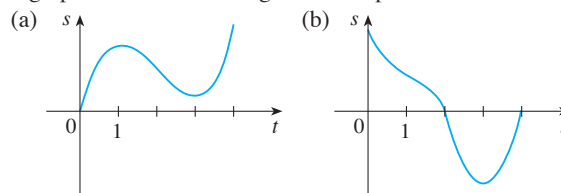
3. $f(t) = \cos(\pi t/4)$, $t \leq 10$

4. $f(t) = te^{-t/2}$

- 5.** Graphs of the *velocity* functions of two particles are shown, where t is measured in seconds. When is each particle speeding up? When is it slowing down? Explain.



- 6.** Graphs of the *position* functions of two particles are shown, where t is measured in seconds. When is each particle speeding up? When is it slowing down? Explain.



7. The height (in meters) of a projectile shot vertically upward from a point 2 m above ground level with an initial velocity of 24.5 m/s is $h = 2 + 24.5t - 4.9t^2$ after t seconds.
- Find the velocity after 2 s and after 4 s.
 - When does the projectile reach its maximum height?
 - What is the maximum height?
 - When does it hit the ground?
 - With what velocity does it hit the ground?
8. If a ball is thrown vertically upward with a velocity of 80 ft/s, then its height after t seconds is $s = 80t - 16t^2$.
- What is the maximum height reached by the ball?
 - What is the velocity of the ball when it is 96 ft above the ground on its way up? On its way down?
9. If a rock is thrown vertically upward from the surface of Mars with velocity 15 m/s, its height after t seconds is $h = 15t - 1.86t^2$.
- What is the velocity of the rock after 2 s?
 - What is the velocity of the rock when its height is 25 m on its way up? On its way down?
10. A particle moves with position function

$$s = t^4 - 4t^3 - 20t^2 + 20t \quad t \geq 0$$

- At what time does the particle have a velocity of 20 m/s?
 - At what time is the acceleration 0? What is the significance of this value of t ?
11. (a) A company makes computer chips from square wafers of silicon. It wants to keep the side length of a wafer very close to 15 mm and it wants to know how the area $A(x)$ of a wafer changes when the side length x changes. Find $A'(15)$ and explain its meaning in this situation.
- (b) Show that the rate of change of the area of a square with respect to its side length is half its perimeter. Try to explain geometrically why this is true by drawing a square whose side length x is increased by an amount Δx . How can you approximate the resulting change in area ΔA if Δx is small?
12. (a) Sodium chlorate crystals are easy to grow in the shape of cubes by allowing a solution of water and sodium chlorate to evaporate slowly. If V is the volume of such a cube with side length x , calculate dV/dx when $x = 3$ mm and explain its meaning.
- (b) Show that the rate of change of the volume of a cube with respect to its edge length is equal to half the surface area of the cube. Explain geometrically why this result is true by arguing by analogy with Exercise 11(b).
13. (a) Find the average rate of change of the area of a circle with respect to its radius r as r changes from
- 2 to 3
 - 2 to 2.5
 - 2 to 2.1
- (b) Find the instantaneous rate of change when $r = 2$.
- (c) Show that the rate of change of the area of a circle with respect to its radius (at any r) is equal to the circumference of the circle. Try to explain geometrically why this is true by drawing a circle whose radius is increased by an

amount Δr . How can you approximate the resulting change in area ΔA if Δr is small?

14. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Find the rate at which the area within the circle is increasing after (a) 1 s, (b) 3 s, and (c) 5 s. What can you conclude?
15. A spherical balloon is being inflated. Find the rate of increase of the surface area ($S = 4\pi r^2$) with respect to the radius r when r is (a) 1 ft, (b) 2 ft, and (c) 3 ft. What conclusion can you make?
16. (a) The volume of a growing spherical cell is $V = \frac{4}{3}\pi r^3$, where the radius r is measured in micrometers ($1 \mu\text{m} = 10^{-6} \text{m}$). Find the average rate of change of V with respect to r when r changes from
- 5 to 8 μm
 - 5 to 6 μm
 - 5 to 5.1 μm
- (b) Find the instantaneous rate of change of V with respect to r when $r = 5 \mu\text{m}$.
- (c) Show that the rate of change of the volume of a sphere with respect to its radius is equal to its surface area. Explain geometrically why this result is true. Argue by analogy with Exercise 13(c).
17. The mass of the part of a metal rod that lies between its left end and a point x meters to the right is $3x^2$ kg. Find the linear density (see Example 2) when x is (a) 1 m, (b) 2 m, and (c) 3 m. Where is the density the highest? The lowest?
18. If a tank holds 5000 gallons of water, which drains from the bottom of the tank in 40 minutes, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V = 5000\left(1 - \frac{1}{40}t\right)^2 \quad 0 \leq t \leq 40$$

Find the rate at which water is draining from the tank after (a) 5 min, (b) 10 min, (c) 20 min, and (d) 40 min. At what time is the water flowing out the fastest? The slowest? Summarize your findings.

19. The quantity of charge Q in coulombs (C) that has passed through a point in a wire up to time t (measured in seconds) is given by $Q(t) = t^3 - 2t^2 + 6t + 2$. Find the current when (a) $t = 0.5$ s and (b) $t = 1$ s. [See Example 3. The unit of current is an ampere ($1 \text{ A} = 1 \text{ C/s}$).] At what time is the current lowest?
20. Newton's Law of Gravitation says that the magnitude F of the force exerted by a body of mass m on a body of mass M is

$$F = \frac{GmM}{r^2}$$

where G is the gravitational constant and r is the distance between the bodies.

- Find dF/dr and explain its meaning. What does the minus sign indicate?
- Suppose it is known that the earth attracts an object with a force that decreases at the rate of 2 N/km when $r = 20,000$ km. How fast does this force change when $r = 10,000$ km?

21. The force F acting on a body with mass m and velocity v is the rate of change of momentum: $F = (d/dt)(mv)$. If m is constant, this becomes $F = ma$, where $a = dv/dt$ is the acceleration. But in the theory of relativity the mass of a particle varies with v as follows: $m = m_0/\sqrt{1 - v^2/c^2}$, where m_0 is the mass of the particle at rest and c is the speed of light. Show that

$$F = \frac{m_0 a}{(1 - v^2/c^2)^{3/2}}$$

22. Some of the highest tides in the world occur in the Bay of Fundy on the Atlantic Coast of Canada. At Hopewell Cape the water depth at low tide is about 2.0 m and at high tide it is about 12.0 m. The natural period of oscillation is a little more than 12 hours and on June 30, 2009, high tide occurred at 6:45 AM. This helps explain the following model for the water depth D (in meters) as a function of the time t (in hours after midnight) on that day:

$$D(t) = 7 + 5 \cos[0.503(t - 6.75)]$$

How fast was the tide rising (or falling) at the following times?

- (a) 3:00 AM (b) 6:00 AM
(c) 9:00 AM (d) Noon
23. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the product of the pressure and the volume remains constant: $PV = C$.
- Find the rate of change of volume with respect to pressure.
 - A sample of gas is in a container at low pressure and is steadily compressed at constant temperature for 10 minutes. Is the volume decreasing more rapidly at the beginning or the end of the 10 minutes? Explain.
 - Prove that the isothermal compressibility (see Example 5) is given by $\beta = 1/P$.
24. If, in Example 4, one molecule of the product C is formed from one molecule of the reactant A and one molecule of the reactant B, and the initial concentrations of A and B have a common value $[A] = [B] = a$ moles/L, then

$$[C] = a^2 kt / (akt + 1)$$

where k is a constant.

- Find the rate of reaction at time t .
 - Show that if $x = [C]$, then
- $$\frac{dx}{dt} = k(a - x)^2$$
- What happens to the concentration as $t \rightarrow \infty$?
 - What happens to the rate of reaction as $t \rightarrow \infty$?
 - What do the results of parts (c) and (d) mean in practical terms?
25. In Example 6 we considered a bacteria population that doubles every hour. Suppose that another population of bacteria triples every hour and starts with 400 bacteria. Find an expression for the number n of bacteria after t hours and use it to estimate the rate of growth of the bacteria population after 2.5 hours.

26. The number of yeast cells in a laboratory culture increases rapidly initially but levels off eventually. The population is modeled by the function

$$n = f(t) = \frac{a}{1 + be^{-0.7t}}$$

where t is measured in hours. At time $t = 0$ the population is 20 cells and is increasing at a rate of 12 cells/hour. Find the values of a and b . According to this model, what happens to the yeast population in the long run?



27. The table gives the population of the world in the 20th century.

Year	Population (in millions)	Year	Population (in millions)
1900	1650	1960	3040
1910	1750	1970	3710
1920	1860	1980	4450
1930	2070	1990	5280
1940	2300	2000	6080
1950	2560		

- Estimate the rate of population growth in 1920 and in 1980 by averaging the slopes of two secant lines.
- Use a graphing calculator or computer to find a cubic function (a third-degree polynomial) that models the data.
- Use your model in part (b) to find a model for the rate of population growth in the 20th century.
- Use part (c) to estimate the rates of growth in 1920 and 1980. Compare with your estimates in part (a).
- Estimate the rate of growth in 1985.



28. The table shows how the average age of first marriage of Japanese women varied in the last half of the 20th century.

t	$A(t)$	t	$A(t)$
1950	23.0	1980	25.2
1955	23.8	1985	25.5
1960	24.4	1990	25.9
1965	24.5	1995	26.3
1970	24.2	2000	27.0
1975	24.7		

- Use a graphing calculator or computer to model these data with a fourth-degree polynomial.
 - Use part (a) to find a model for $A'(t)$.
 - Estimate the rate of change of marriage age for women in 1990.
 - Graph the data points and the models for A and A' .
29. Refer to the law of laminar flow given in Example 7. Consider a blood vessel with radius 0.01 cm, length 3 cm, pressure difference 3000 dynes/cm², and viscosity $\eta = 0.027$.
- Find the velocity of the blood along the centerline $r = 0$, at radius $r = 0.005$ cm, and at the wall $r = R = 0.01$ cm.

- (b) Find the velocity gradient at $r = 0$, $r = 0.005$, and $r = 0.01$.
- (c) Where is the velocity the greatest? Where is the velocity changing most?
30. The frequency of vibrations of a vibrating violin string is given by

$$f = \frac{1}{2L} \sqrt{\frac{T}{\rho}}$$

where L is the length of the string, T is its tension, and ρ is its linear density. [See Chapter 11 in D. E. Hall, *Musical Acoustics*, 3rd ed. (Pacific Grove, CA, 2002).]

- (a) Find the rate of change of the frequency with respect to
- the length (when T and ρ are constant),
 - the tension (when L and ρ are constant), and
 - the linear density (when L and T are constant).
- (b) The pitch of a note (how high or low the note sounds) is determined by the frequency f . (The higher the frequency, the higher the pitch.) Use the signs of the derivatives in part (a) to determine what happens to the pitch of a note
- when the effective length of a string is decreased by placing a finger on the string so a shorter portion of the string vibrates,
 - when the tension is increased by turning a tuning peg,
 - when the linear density is increased by switching to another string.

31. The cost, in dollars, of producing x yards of a certain fabric is

$$C(x) = 1200 + 12x - 0.1x^2 + 0.0005x^3$$

- (a) Find the marginal cost function.
- (b) Find $C'(200)$ and explain its meaning. What does it predict?
- (c) Compare $C'(200)$ with the cost of manufacturing the 201st yard of fabric.
32. The cost function for production of a commodity is

$$C(x) = 339 + 25x - 0.09x^2 + 0.0004x^3$$

- (a) Find and interpret $C'(100)$.
- (b) Compare $C'(100)$ with the cost of producing the 101st item.
33. If $p(x)$ is the total value of the production when there are x workers in a plant, then the *average productivity* of the workforce at the plant is

$$A(x) = \frac{p(x)}{x}$$

- (a) Find $A'(x)$. Why does the company want to hire more workers if $A'(x) > 0$?
- (b) Show that $A'(x) > 0$ if $p'(x)$ is greater than the average productivity.
34. If R denotes the reaction of the body to some stimulus of strength x , the *sensitivity* S is defined to be the rate of change

of the reaction with respect to x . A particular example is that when the brightness x of a light source is increased, the eye reacts by decreasing the area R of the pupil. The experimental formula

$$R = \frac{40 + 24x^{0.4}}{1 + 4x^{0.4}}$$

has been used to model the dependence of R on x when R is measured in square millimeters and x is measured in appropriate units of brightness.



- (a) Find the sensitivity.
- (b) Illustrate part (a) by graphing both R and S as functions of x . Comment on the values of R and S at low levels of brightness. Is this what you would expect?
35. The gas law for an ideal gas at absolute temperature T (in kelvins), pressure P (in atmospheres), and volume V (in liters) is $PV = nRT$, where n is the number of moles of the gas and $R = 0.0821$ is the gas constant. Suppose that, at a certain instant, $P = 8.0$ atm and is increasing at a rate of 0.10 atm/min and $V = 10$ L and is decreasing at a rate of 0.15 L/min. Find the rate of change of T with respect to time at that instant if $n = 10$ mol.

36. In a fish farm, a population of fish is introduced into a pond and harvested regularly. A model for the rate of change of the fish population is given by the equation

$$\frac{dP}{dt} = r_0 \left(1 - \frac{P(t)}{P_c} \right) P(t) - \beta P(t)$$

where r_0 is the birth rate of the fish, P_c is the maximum population that the pond can sustain (called the *carrying capacity*), and β is the percentage of the population that is harvested.

- (a) What value of dP/dt corresponds to a stable population?
- (b) If the pond can sustain 10,000 fish, the birth rate is 5%, and the harvesting rate is 4%, find the stable population level.
- (c) What happens if β is raised to 5%?
37. In the study of ecosystems, *predator-prey models* are often used to study the interaction between species. Consider populations of tundra wolves, given by $W(t)$, and caribou, given by $C(t)$, in northern Canada. The interaction has been modeled by the equations

$$\frac{dC}{dt} = aC - bCW \quad \frac{dW}{dt} = -cW + dCW$$

- (a) What values of dC/dt and dW/dt correspond to stable populations?
- (b) How would the statement “The caribou go extinct” be represented mathematically?
- (c) Suppose that $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$. Find all population pairs (C, W) that lead to stable populations. According to this model, is it possible for the two species to live in balance or will one or both species become extinct?

3.8 Exponential Growth and Decay

In many natural phenomena, quantities grow or decay at a rate proportional to their size. For instance, if $y = f(t)$ is the number of individuals in a population of animals or bacteria at time t , then it seems reasonable to expect that the rate of growth $f'(t)$ is proportional to the population $f(t)$; that is, $f'(t) = kf(t)$ for some constant k . Indeed, under ideal conditions (unlimited environment, adequate nutrition, immunity to disease) the mathematical model given by the equation $f'(t) = kf(t)$ predicts what actually happens fairly accurately. Another example occurs in nuclear physics where the mass of a radioactive substance decays at a rate proportional to the mass. In chemistry, the rate of a unimolecular first-order reaction is proportional to the concentration of the substance. In finance, the value of a savings account with continuously compounded interest increases at a rate proportional to that value.

In general, if $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then

1

$$\frac{dy}{dt} = ky$$

where k is a constant. Equation 1 is sometimes called the **law of natural growth** (if $k > 0$) or the **law of natural decay** (if $k < 0$). It is called a **differential equation** because it involves an unknown function y and its derivative dy/dt .

It's not hard to think of a solution of Equation 1. This equation asks us to find a function whose derivative is a constant multiple of itself. We have met such functions in this chapter. Any exponential function of the form $y(t) = Ce^{kt}$, where C is a constant, satisfies

$$y'(t) = C(ke^{kt}) = k(Ce^{kt}) = ky(t)$$

We will see in Section 9.4 that *any* function that satisfies $dy/dt = ky$ must be of the form $y = Ce^{kt}$. To see the significance of the constant C , we observe that

$$y(0) = Ce^{k \cdot 0} = C$$

Therefore C is the initial value of the function.

2 Theorem The only solutions of the differential equation $dy/dt = ky$ are the exponential functions

$$y(t) = y(0)e^{kt}$$

Population Growth

What is the significance of the proportionality constant k ? In the context of population growth, where $P(t)$ is the size of a population at time t , we can write

3

$$\frac{dP}{dt} = kP \quad \text{or} \quad \frac{1}{P} \frac{dP}{dt} = k$$

The quantity

$$\frac{1}{P} \frac{dP}{dt}$$

is the growth rate divided by the population size; it is called the **relative growth rate**. According to [3], instead of saying “the growth rate is proportional to population size”

we could say “the relative growth rate is constant.” Then [2] says that a population with constant relative growth rate must grow exponentially. Notice that the relative growth rate k appears as the coefficient of t in the exponential function Ce^{kt} . For instance, if

$$\frac{dP}{dt} = 0.02P$$

and t is measured in years, then the relative growth rate is $k = 0.02$ and the population grows at a relative rate of 2% per year. If the population at time 0 is P_0 , then the expression for the population is

$$P(t) = P_0 e^{0.02t}$$

V EXAMPLE 1 Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the relative growth rate? Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

SOLUTION We measure the time t in years and let $t = 0$ in the year 1950. We measure the population $P(t)$ in millions of people. Then $P(0) = 2560$ and $P(10) = 3040$. Since we are assuming that $dP/dt = kP$, Theorem 2 gives

$$P(t) = P(0)e^{kt} = 2560e^{kt}$$

$$P(10) = 2560e^{10k} = 3040$$

$$k = \frac{1}{10} \ln \frac{3040}{2560} \approx 0.017185$$

The relative growth rate is about 1.7% per year and the model is

$$P(t) = 2560e^{0.017185t}$$

We estimate that the world population in 1993 was

$$P(43) = 2560e^{0.017185(43)} \approx 5360 \text{ million}$$

The model predicts that the population in 2020 will be

$$P(70) = 2560e^{0.017185(70)} \approx 8524 \text{ million}$$

The graph in Figure 1 shows that the model is fairly accurate to the end of the 20th century (the dots represent the actual population), so the estimate for 1993 is quite reliable. But the prediction for 2020 is riskier.

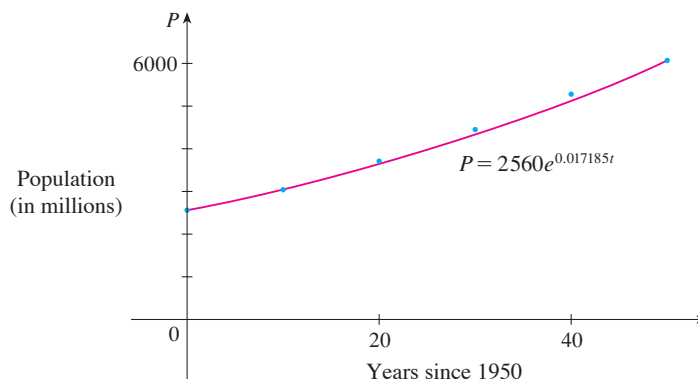


FIGURE 1

A model for world population growth in the second half of the 20th century

Radioactive Decay

Radioactive substances decay by spontaneously emitting radiation. If $m(t)$ is the mass remaining from an initial mass m_0 of the substance after time t , then the relative decay rate

$$-\frac{1}{m} \frac{dm}{dt}$$

has been found experimentally to be constant. (Since dm/dt is negative, the relative decay rate is positive.) It follows that

$$\frac{dm}{dt} = km$$

where k is a negative constant. In other words, radioactive substances decay at a rate proportional to the remaining mass. This means that we can use [\[2\]](#) to show that the mass decays exponentially:

$$m(t) = m_0 e^{kt}$$

Physicists express the rate of decay in terms of **half-life**, the time required for half of any given quantity to decay.

V EXAMPLE 2 The half-life of radium-226 is 1590 years.

- (a) A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of the sample that remains after t years.
- (b) Find the mass after 1000 years correct to the nearest milligram.
- (c) When will the mass be reduced to 30 mg?

SOLUTION

(a) Let $m(t)$ be the mass of radium-226 (in milligrams) that remains after t years. Then $dm/dt = km$ and $y(0) = 100$, so [\[2\]](#) gives

$$m(t) = m(0)e^{kt} = 100e^{kt}$$

In order to determine the value of k , we use the fact that $y(1590) = \frac{1}{2}(100)$. Thus

$$100e^{1590k} = 50 \quad \text{so} \quad e^{1590k} = \frac{1}{2}$$

and

$$1590k = \ln \frac{1}{2} = -\ln 2$$

$$k = -\frac{\ln 2}{1590}$$

Therefore

$$m(t) = 100e^{-(\ln 2)t/1590}$$

We could use the fact that $e^{\ln 2} = 2$ to write the expression for $m(t)$ in the alternative form

$$m(t) = 100 \times 2^{-t/1590}$$

(b) The mass after 1000 years is

$$m(1000) = 100e^{-(\ln 2)1000/1590} \approx 65 \text{ mg}$$

(c) We want to find the value of t such that $m(t) = 30$, that is,

$$100e^{-(\ln 2)t/1590} = 30 \quad \text{or} \quad e^{-(\ln 2)t/1590} = 0.3$$

We solve this equation for t by taking the natural logarithm of both sides:

$$-\frac{\ln 2}{1590}t = \ln 0.3$$

Thus

$$t = -1590 \frac{\ln 0.3}{\ln 2} \approx 2762 \text{ years}$$

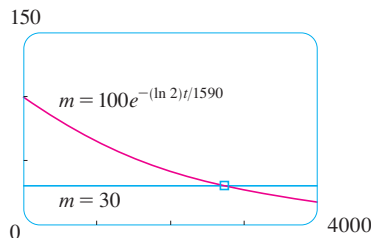


FIGURE 2

As a check on our work in Example 2, we use a graphing device to draw the graph of $m(t)$ in Figure 2 together with the horizontal line $m = 30$. These curves intersect when $t \approx 2800$, and this agrees with the answer to part (c).

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. (This law also applies to warming.) If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings, then we can formulate Newton's Law of Cooling as a differential equation:

$$\frac{dT}{dt} = k(T - T_s)$$

where k is a constant. This equation is not quite the same as Equation 1, so we make the change of variable $y(t) = T(t) - T_s$. Because T_s is constant, we have $y'(t) = T'(t)$ and so the equation becomes

$$\frac{dy}{dt} = ky$$

We can then use [2] to find an expression for y , from which we can find T .

EXAMPLE 3 A bottle of soda pop at room temperature (72°F) is placed in a refrigerator where the temperature is 44°F . After half an hour the soda pop has cooled to 61°F .

- What is the temperature of the soda pop after another half hour?
- How long does it take for the soda pop to cool to 50°F ?

SOLUTION

(a) Let $T(t)$ be the temperature of the soda after t minutes. The surrounding temperature is $T_s = 44^\circ\text{F}$, so Newton's Law of Cooling states that

$$\frac{dT}{dt} = k(T - 44)$$

If we let $y = T - 44$, then $y(0) = T(0) - 44 = 72 - 44 = 28$, so y satisfies

$$\frac{dy}{dt} = ky \quad y(0) = 28$$

and by [2] we have

$$y(t) = y(0)e^{kt} = 28e^{kt}$$

We are given that $T(30) = 61$, so $y(30) = 61 - 44 = 17$ and

$$28e^{30k} = 17 \quad e^{30k} = \frac{17}{28}$$

Taking logarithms, we have

$$k = \frac{\ln\left(\frac{17}{28}\right)}{30} \approx -0.01663$$

Thus

$$y(t) = 28e^{-0.01663t}$$

$$T(t) = 44 + 28e^{-0.01663t}$$

$$T(60) = 44 + 28e^{-0.01663(60)} \approx 54.3$$

So after another half hour the pop has cooled to about 54°F.

(b) We have $T(t) = 50$ when

$$44 + 28e^{-0.01663t} = 50$$

$$e^{-0.01663t} = \frac{6}{28}$$

$$t = \frac{\ln\left(\frac{6}{28}\right)}{-0.01663} \approx 92.6$$

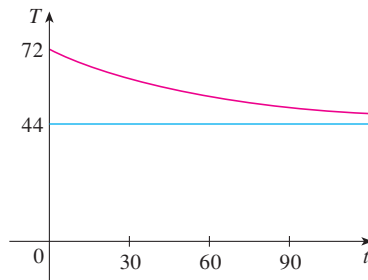


FIGURE 3

The pop cools to 50°F after about 1 hour 33 minutes.

Notice that in Example 3, we have

$$\lim_{t \rightarrow \infty} T(t) = \lim_{t \rightarrow \infty} (44 + 28e^{-0.01663t}) = 44 + 28 \cdot 0 = 44$$

which is to be expected. The graph of the temperature function is shown in Figure 3.

Continuously Compounded Interest

EXAMPLE 4 If \$1000 is invested at 6% interest, compounded annually, then after 1 year the investment is worth $\$1000(1.06) = \1060 , after 2 years it's worth $\$[1000(1.06)]1.06 = \1123.60 , and after t years it's worth $\$1000(1.06)^t$. In general, if an amount A_0 is invested at an interest rate r ($r = 0.06$ in this example), then after t years it's worth $A_0(1 + r)^t$. Usually, however, interest is compounded more frequently, say, n times a year. Then in each compounding period the interest rate is r/n and there are nt compounding periods in t years, so the value of the investment is

$$A_0 \left(1 + \frac{r}{n}\right)^{nt}$$

For instance, after 3 years at 6% interest a \$1000 investment will be worth

$$\$1000(1.06)^3 = \$1191.02 \quad \text{with annual compounding}$$

$$\$1000(1.03)^6 = \$1194.05 \quad \text{with semiannual compounding}$$

$$\$1000(1.015)^{12} = \$1195.62 \quad \text{with quarterly compounding}$$

$$\$1000(1.005)^{36} = \$1196.68 \quad \text{with monthly compounding}$$

$$\$1000 \left(1 + \frac{0.06}{365}\right)^{365 \cdot 3} = \$1197.20 \quad \text{with daily compounding}$$

You can see that the interest paid increases as the number of compounding periods (n) increases. If we let $n \rightarrow \infty$, then we will be compounding the interest **continuously** and the value of the investment will be

$$\begin{aligned} A(t) &= \lim_{n \rightarrow \infty} A_0 \left(1 + \frac{r}{n} \right)^{nt} \\ &= \lim_{n \rightarrow \infty} A_0 \left[\left(1 + \frac{r}{n} \right)^{n/r} \right]^{rt} \\ &= A_0 \left[\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n} \right)^{n/r} \right]^{rt} \\ &= A_0 \left[\lim_{m \rightarrow \infty} \left(1 + \frac{1}{m} \right)^m \right]^{rt} \quad (\text{where } m = n/r) \end{aligned}$$

But the limit in this expression is equal to the number e (see Equation 3.6.6). So with continuous compounding of interest at interest rate r , the amount after t years is

$$A(t) = A_0 e^{rt}$$

If we differentiate this equation, we get

$$\frac{dA}{dt} = rA_0 e^{rt} = rA(t)$$

which says that, with continuous compounding of interest, the rate of increase of an investment is proportional to its size.

Returning to the example of \$1000 invested for 3 years at 6% interest, we see that with continuous compounding of interest the value of the investment will be

$$A(3) = \$1000e^{(0.06)3} = \$1197.22$$

Notice how close this is to the amount we calculated for daily compounding, \$1197.20. But the amount is easier to compute if we use continuous compounding.

3.8 Exercises

- A population of protozoa develops with a constant relative growth rate of 0.7944 per member per day. On day zero the population consists of two members. Find the population size after six days.
- A common inhabitant of human intestines is the bacterium *Escherichia coli*. A cell of this bacterium in a nutrient-broth medium divides into two cells every 20 minutes. The initial population of a culture is 60 cells.
 - Find the relative growth rate.
 - Find an expression for the number of cells after t hours.
 - Find the number of cells after 8 hours.
 - Find the rate of growth after 8 hours.
 - When will the population reach 20,000 cells?
- A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420.
 - Find an expression for the number of bacteria after t hours.
 - Find the number of bacteria after 3 hours.
 - Find the rate of growth after 3 hours.
 - When will the population reach 10,000?

4. A bacteria culture grows with constant relative growth rate. The bacteria count was 400 after 2 hours and 25,600 after 6 hours.
- What is the relative growth rate? Express your answer as a percentage.
 - What was the initial size of the culture?
 - Find an expression for the number of bacteria after t hours.
 - Find the number of cells after 4.5 hours.
 - Find the rate of growth after 4.5 hours.
 - When will the population reach 50,000?
5. The table gives estimates of the world population, in millions, from 1750 to 2000.
- Use the exponential model and the population figures for 1750 and 1800 to predict the world population in 1900 and 1950. Compare with the actual figures.
 - Use the exponential model and the population figures for 1850 and 1900 to predict the world population in 1950. Compare with the actual population.
 - Use the exponential model and the population figures for 1900 and 1950 to predict the world population in 2000. Compare with the actual population and try to explain the discrepancy.

Year	Population	Year	Population
1750	790	1900	1650
1800	980	1950	2560
1850	1260	2000	6080

6. The table gives the population of India, in millions, for the second half of the 20th century.

Year	Population
1951	361
1961	439
1971	548
1981	683
1991	846
2001	1029

- Use the exponential model and the census figures for 1951 and 1961 to predict the population in 2001. Compare with the actual figure.
- Use the exponential model and the census figures for 1961 and 1981 to predict the population in 2001. Compare with the actual population. Then use this model to predict the population in the years 2010 and 2020.
- Graph both of the exponential functions in parts (a) and (b) together with a plot of the actual population. Are these models reasonable ones?

7. Experiments show that if the chemical reaction



takes place at 45°C, the rate of reaction of dinitrogen pent-

oxide is proportional to its concentration as follows:

$$-\frac{d[\text{N}_2\text{O}_5]}{dt} = 0.0005[\text{N}_2\text{O}_5]$$

(See Example 4 in Section 3.7.)

- Find an expression for the concentration $[\text{N}_2\text{O}_5]$ after t seconds if the initial concentration is C .
 - How long will the reaction take to reduce the concentration of N_2O_5 to 90% of its original value?
8. Strontium-90 has a half-life of 28 days.
- A sample has a mass of 50 mg initially. Find a formula for the mass remaining after t days.
 - Find the mass remaining after 40 days.
 - How long does it take the sample to decay to a mass of 2 mg?
 - Sketch the graph of the mass function.
9. The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample.
- Find the mass that remains after t years.
 - How much of the sample remains after 100 years?
 - After how long will only 1 mg remain?
10. A sample of tritium-3 decayed to 94.5% of its original amount after a year.
- What is the half-life of tritium-3?
 - How long would it take the sample to decay to 20% of its original amount?
11. Scientists can determine the age of ancient objects by the method of *radiocarbon dating*. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, ^{14}C , with a half-life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates ^{14}C through food chains. When a plant or animal dies, it stops replacing its carbon and the amount of ^{14}C begins to decrease through radioactive decay. Therefore the level of radioactivity must also decay exponentially.
- A parchment fragment was discovered that had about 74% as much ^{14}C radioactivity as does plant material on the earth today. Estimate the age of the parchment.
12. A curve passes through the point $(0, 5)$ and has the property that the slope of the curve at every point P is twice the y -coordinate of P . What is the equation of the curve?
13. A roast turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F.
- If the temperature of the turkey is 150°F after half an hour, what is the temperature after 45 minutes?
 - When will the turkey have cooled to 100°F?
14. In a murder investigation, the temperature of the corpse was 32.5°C at 1:30 PM and 30.3°C an hour later. Normal body temperature is 37.0°C and the temperature of the surroundings was 20.0°C. When did the murder take place?

15. When a cold drink is taken from a refrigerator, its temperature is 5°C . After 25 minutes in a 20°C room its temperature has increased to 10°C .
 (a) What is the temperature of the drink after 50 minutes?
 (b) When will its temperature be 15°C ?
16. A freshly brewed cup of coffee has temperature 95°C in a 20°C room. When its temperature is 70°C , it is cooling at a rate of 1°C per minute. When does this occur?
17. The rate of change of atmospheric pressure P with respect to altitude h is proportional to P , provided that the temperature is constant. At 15°C the pressure is 101.3 kPa at sea level and 87.14 kPa at $h = 1000$ m.
 (a) What is the pressure at an altitude of 3000 m?
 (b) What is the pressure at the top of Mount McKinley, at an altitude of 6187 m?
18. (a) If \$1000 is borrowed at 8% interest, find the amounts due at the end of 3 years if the interest is compounded
 (i) annually, (ii) quarterly, (iii) monthly, (iv) weekly, (v) daily, (vi) hourly, and (vii) continuously.
 (b) Suppose \$1000 is borrowed and the interest is compounded continuously. If $A(t)$ is the amount due after t years, where $0 \leq t \leq 3$, graph $A(t)$ for each of the interest rates 6%, 8%, and 10% on a common screen.
19. (a) If \$3000 is invested at 5% interest, find the value of the investment at the end of 5 years if the interest is compounded (i) annually, (ii) semiannually, (iii) monthly, (iv) weekly, (v) daily, and (vi) continuously.
 (b) If $A(t)$ is the amount of the investment at time t for the case of continuous compounding, write a differential equation and an initial condition satisfied by $A(t)$.
20. (a) How long will it take an investment to double in value if the interest rate is 6% compounded continuously?
 (b) What is the equivalent annual interest rate?

3.9 Related Rates

If we are pumping air into a balloon, both the volume and the radius of the balloon are increasing and their rates of increase are related to each other. But it is much easier to measure directly the rate of increase of the volume than the rate of increase of the radius.

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

V EXAMPLE 1 Air is being pumped into a spherical balloon so that its volume increases at a rate of $100 \text{ cm}^3/\text{s}$. How fast is the radius of the balloon increasing when the diameter is 50 cm?

SOLUTION We start by identifying two things:

the *given information*:

the rate of increase of the volume of air is $100 \text{ cm}^3/\text{s}$

and the *unknown*:

the rate of increase of the radius when the diameter is 50 cm

In order to express these quantities mathematically, we introduce some suggestive notation:

Let V be the volume of the balloon and let r be its radius.

The key thing to remember is that rates of change are derivatives. In this problem, the volume and the radius are both functions of the time t . The rate of increase of the volume with respect to time is the derivative dV/dt , and the rate of increase of the radius is dr/dt . We can therefore restate the given and the unknown as follows:

$$\text{Given:} \quad \frac{dV}{dt} = 100 \text{ cm}^3/\text{s}$$

$$\text{Unknown:} \quad \frac{dr}{dt} \quad \text{when } r = 25 \text{ cm}$$

PS According to the Principles of Problem Solving discussed on page 75, the first step is to understand the problem. This includes reading the problem carefully, identifying the given and the unknown, and introducing suitable notation.

PS The second stage of problem solving is to think of a plan for connecting the given and the unknown.

In order to connect dV/dt and dr/dt , we first relate V and r by the formula for the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

In order to use the given information, we differentiate each side of this equation with respect to t . To differentiate the right side, we need to use the Chain Rule:

$$\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now we solve for the unknown quantity:

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \frac{dV}{dt}$$

Notice that, although dV/dt is constant, dr/dt is *not* constant.

If we put $r = 25$ and $dV/dt = 100$ in this equation, we obtain

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2} 100 = \frac{1}{25\pi}$$

The radius of the balloon is increasing at the rate of $1/(25\pi) \approx 0.0127$ cm/s.

EXAMPLE 2 A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

SOLUTION We first draw a diagram and label it as in Figure 1. Let x feet be the distance from the bottom of the ladder to the wall and y feet the distance from the top of the ladder to the ground. Note that x and y are both functions of t (time, measured in seconds).

We are given that $dx/dt = 1$ ft/s and we are asked to find dy/dt when $x = 6$ ft (see Figure 2). In this problem, the relationship between x and y is given by the Pythagorean Theorem:

$$x^2 + y^2 = 100$$

Differentiating each side with respect to t using the Chain Rule, we have

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

and solving this equation for the desired rate, we obtain

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$$

When $x = 6$, the Pythagorean Theorem gives $y = 8$ and so, substituting these values and $dx/dt = 1$, we have

$$\frac{dy}{dt} = -\frac{6}{8}(1) = -\frac{3}{4} \text{ ft/s}$$

The fact that dy/dt is negative means that the distance from the top of the ladder to the ground is *decreasing* at a rate of $\frac{3}{4}$ ft/s. In other words, the top of the ladder is sliding down the wall at a rate of $\frac{3}{4}$ ft/s.

EXAMPLE 3 A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of $2 \text{ m}^3/\text{min}$, find the rate at which the water level is rising when the water is 3 m deep.

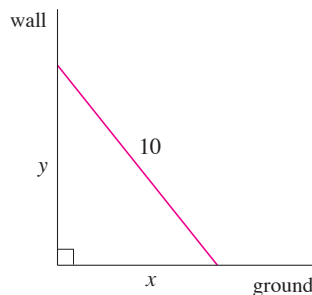


FIGURE 1

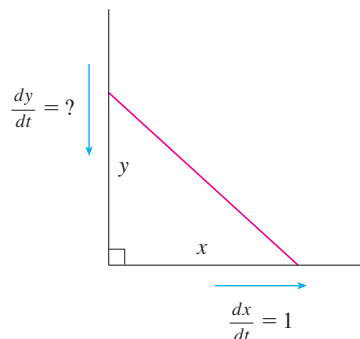


FIGURE 2

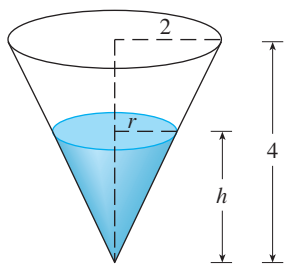


FIGURE 3

SOLUTION We first sketch the cone and label it as in Figure 3. Let V , r , and h be the volume of the water, the radius of the surface, and the height of the water at time t , where t is measured in minutes.

We are given that $dV/dt = 2 \text{ m}^3/\text{min}$ and we are asked to find dh/dt when h is 3 m. The quantities V and h are related by the equation

$$V = \frac{1}{3}\pi r^2 h$$

but it is very useful to express V as a function of h alone. In order to eliminate r , we use the similar triangles in Figure 3 to write

$$\frac{r}{h} = \frac{2}{4} \quad r = \frac{h}{2}$$

and the expression for V becomes

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Now we can differentiate each side with respect to t :

$$\frac{dV}{dt} = \frac{\pi}{4}h^2 \frac{dh}{dt}$$

so

$$\frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt}$$

Substituting $h = 3 \text{ m}$ and $dV/dt = 2 \text{ m}^3/\text{min}$, we have

$$\frac{dh}{dt} = \frac{4}{\pi(3)^2} \cdot 2 = \frac{8}{9\pi}$$

The water level is rising at a rate of $8/(9\pi) \approx 0.28 \text{ m/min}$.

PS Look back: What have we learned from Examples 1–3 that will help us solve future problems?

WARNING A common error is to substitute the given numerical information (for quantities that vary with time) too early. This should be done only *after* the differentiation. (Step 7 follows Step 6.) For instance, in Example 3 we dealt with general values of h until we finally substituted $h = 3$ at the last stage. (If we had put $h = 3$ earlier, we would have gotten $dV/dt = 0$, which is clearly wrong.)

Problem Solving Strategy It is useful to recall some of the problem-solving principles from page 75 and adapt them to related rates in light of our experience in Examples 1–3:

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution (as in Example 3).
6. Use the Chain Rule to differentiate both sides of the equation with respect to t .
7. Substitute the given information into the resulting equation and solve for the unknown rate.

The following examples are further illustrations of the strategy.

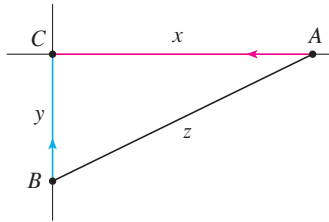


FIGURE 4

V EXAMPLE 4 Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

SOLUTION We draw Figure 4, where C is the intersection of the roads. At a given time t , let x be the distance from car A to C , let y be the distance from car B to C , and let z be the distance between the cars, where x , y , and z are measured in miles.

We are given that $dx/dt = -50$ mi/h and $dy/dt = -60$ mi/h. (The derivatives are negative because x and y are decreasing.) We are asked to find dz/dt . The equation that relates x , y , and z is given by the Pythagorean Theorem:

$$z^2 = x^2 + y^2$$

Differentiating each side with respect to t , we have

$$\begin{aligned} 2z \frac{dz}{dt} &= 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \\ \frac{dz}{dt} &= \frac{1}{z} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right) \end{aligned}$$

When $x = 0.3$ mi and $y = 0.4$ mi, the Pythagorean Theorem gives $z = 0.5$ mi, so

$$\begin{aligned} \frac{dz}{dt} &= \frac{1}{0.5} [0.3(-50) + 0.4(-60)] \\ &= -78 \text{ mi/h} \end{aligned}$$

The cars are approaching each other at a rate of 78 mi/h.

V EXAMPLE 5 A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

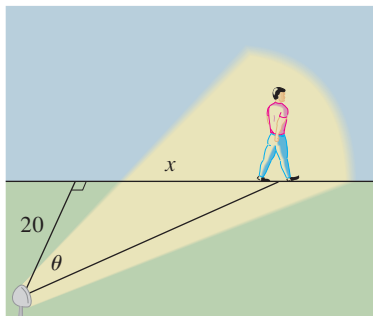


FIGURE 5

SOLUTION We draw Figure 5 and let x be the distance from the man to the point on the path closest to the searchlight. We let θ be the angle between the beam of the searchlight and the perpendicular to the path.

We are given that $dx/dt = 4$ ft/s and are asked to find $d\theta/dt$ when $x = 15$. The equation that relates x and θ can be written from Figure 5:

$$\frac{x}{20} = \tan \theta \quad x = 20 \tan \theta$$

Differentiating each side with respect to t , we get

$$\frac{dx}{dt} = 20 \sec^2 \theta \frac{d\theta}{dt}$$

so

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{1}{20} \cos^2 \theta \frac{dx}{dt} \\ &= \frac{1}{20} \cos^2 \theta (4) = \frac{1}{5} \cos^2 \theta \end{aligned}$$

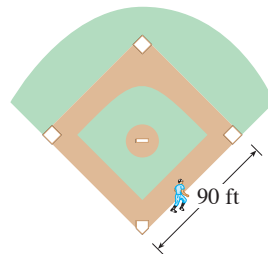
When $x = 15$, the length of the beam is 25, so $\cos \theta = \frac{4}{5}$ and

$$\frac{d\theta}{dt} = \frac{1}{5} \left(\frac{4}{5} \right)^2 = \frac{16}{125} = 0.128$$

The searchlight is rotating at a rate of 0.128 rad/s.

3.9 Exercises

- If V is the volume of a cube with edge length x and the cube expands as time passes, find dV/dt in terms of dx/dt .
- (a) If A is the area of a circle with radius r and the circle expands as time passes, find dA/dt in terms of dr/dt .
(b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1 m/s, how fast is the area of the spill increasing when the radius is 30 m?
- Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm^2 ?
- The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?
- A cylindrical tank with radius 5 m is being filled with water at a rate of $3 \text{ m}^3/\text{min}$. How fast is the height of the water increasing?
- The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80 mm?
- Suppose $y = \sqrt{2x + 1}$, where x and y are functions of t .
(a) If $dx/dt = 3$, find dy/dt when $x = 4$.
(b) If $dy/dt = 5$, find dx/dt when $x = 12$.
- Suppose $4x^2 + 9y^2 = 36$, where x and y are functions of t .
(a) If $dy/dt = \frac{1}{3}$, find dx/dt when $x = 2$ and $y = \frac{2}{3}\sqrt{5}$.
(b) If $dx/dt = 3$, find dy/dt when $x = -2$ and $y = \frac{2}{3}\sqrt{5}$.
- If $x^2 + y^2 + z^2 = 9$, $dx/dt = 5$, and $dy/dt = 4$, find dz/dt when $(x, y, z) = (2, 2, 1)$.
- A particle is moving along a hyperbola $xy = 8$. As it reaches the point $(4, 2)$, the y -coordinate is decreasing at a rate of 3 cm/s. How fast is the x -coordinate of the point changing at that instant?
- which the distance from the plane to the station is increasing when it is 2 mi away from the station.
- If a snowball melts so that its surface area decreases at a rate of $1 \text{ cm}^2/\text{min}$, find the rate at which the diameter decreases when the diameter is 10 cm.
- A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?
- At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 PM?
- Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?
- A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?
- A man starts walking north at 4 ft/s from a point P . Five minutes later a woman starts walking south at 5 ft/s from a point 500 ft due east of P . At what rate are the people moving apart 15 min after the woman starts walking?
- A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/s.
(a) At what rate is his distance from second base decreasing when he is halfway to first base?
(b) At what rate is his distance from third base increasing at the same moment?



11–14

- What quantities are given in the problem?
- What is the unknown?
- Draw a picture of the situation for any time t .
- Write an equation that relates the quantities.
- Finish solving the problem.

- A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at

- The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of



Graphing calculator or computer required

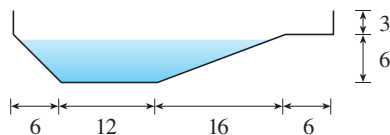
1. Homework Hints available at stewartcalculus.com

2 cm²/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm²?

20. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?

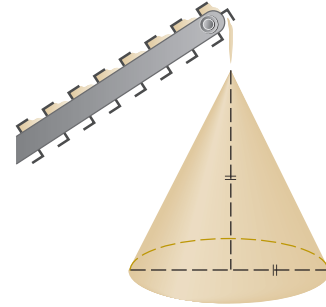


21. At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 PM?
22. A particle moves along the curve $y = 2 \sin(\pi x/2)$. As the particle passes through the point $(\frac{1}{3}, 1)$, its x -coordinate increases at a rate of $\sqrt{10}$ cm/s. How fast is the distance from the particle to the origin changing at this instant?
23. Water is leaking out of an inverted conical tank at a rate of 10,000 cm³/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.
24. A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of 12 ft³/min, how fast is the water level rising when the water is 6 inches deep?
25. A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has height 50 cm. If the trough is being filled with water at the rate of 0.2 m³/min, how fast is the water level rising when the water is 30 cm deep?
26. A swimming pool is 20 ft wide, 40 ft long, 3 ft deep at the shallow end, and 9 ft deep at its deepest point. A cross-section is shown in the figure. If the pool is being filled at a rate of 0.8 ft³/min, how fast is the water level rising when the depth at the deepest point is 5 ft?



27. Gravel is being dumped from a conveyor belt at a rate of 30 ft³/min, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always

equal. How fast is the height of the pile increasing when the pile is 10 ft high?

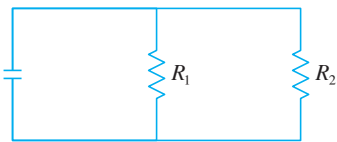


28. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?
29. Two sides of a triangle are 4 m and 5 m in length and the angle between them is increasing at a rate of 0.06 rad/s. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\pi/3$.
30. How fast is the angle between the ladder and the ground changing in Example 2 when the bottom of the ladder is 6 ft from the wall?
31. The top of a ladder slides down a vertical wall at a rate of 0.15 m/s. At the moment when the bottom of the ladder is 3 m from the wall, it slides away from the wall at a rate of 0.2 m/s. How long is the ladder?
32. A faucet is filling a hemispherical basin of diameter 60 cm with water at a rate of 2 L/min. Find the rate at which the water is rising in the basin when it is half full. [Use the following facts: 1 L is 1000 cm³. The volume of the portion of a sphere with radius r from the bottom to a height h is $V = \pi(rh^2 - \frac{1}{3}h^3)$, as we will show in Chapter 6.]
33. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = C$, where C is a constant. Suppose that at a certain instant the volume is 600 cm³, the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume decreasing at this instant?
34. When air expands adiabatically (without gaining or losing heat), its pressure P and volume V are related by the equation $PV^{1.4} = C$, where C is a constant. Suppose that at a certain instant the volume is 400 cm³ and the pressure is 80 kPa and is decreasing at a rate of 10 kPa/min. At what rate is the volume increasing at this instant?
35. If two resistors with resistances R_1 and R_2 are connected in parallel, as in the figure, then the total resistance R , measured in ohms (Ω), is given by

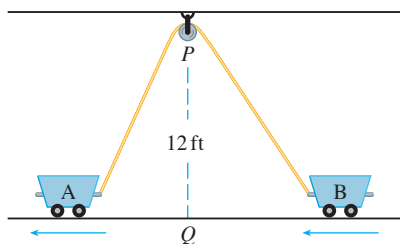
$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

If R_1 and R_2 are increasing at rates of 0.3 Ω /s and 0.2 Ω /s,

respectively, how fast is R changing when $R_1 = 80\ \Omega$ and $R_2 = 100\ \Omega$?



36. Brain weight B as a function of body weight W in fish has been modeled by the power function $B = 0.007W^{2/3}$, where B and W are measured in grams. A model for body weight as a function of body length L (measured in centimeters) is $W = 0.12L^{2.53}$. If, over 10 million years, the average length of a certain species of fish evolved from 15 cm to 20 cm at a constant rate, how fast was this species' brain growing when the average length was 18 cm?
37. Two sides of a triangle have lengths 12 m and 15 m. The angle between them is increasing at a rate of $2^\circ/\text{min}$. How fast is the length of the third side increasing when the angle between the sides of fixed length is 60° ?
38. Two carts, A and B, are connected by a rope 39 ft long that passes over a pulley P (see the figure). The point Q is on the floor 12 ft directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 2 ft/s. How fast is cart B moving toward Q at the instant when cart A is 5 ft from Q ?



39. A television camera is positioned 4000 ft from the base of a rocket launching pad. The angle of elevation of the camera has

to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Let's assume the rocket rises vertically and its speed is 600 ft/s when it has risen 3000 ft.

- (a) How fast is the distance from the television camera to the rocket changing at that moment?
- (b) If the television camera is always kept aimed at the rocket, how fast is the camera's angle of elevation changing at that same moment?
40. A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P ?
41. A plane flies horizontally at an altitude of 5 km and passes directly over a tracking telescope on the ground. When the angle of elevation is $\pi/3$, this angle is decreasing at a rate of $\pi/6$ rad/min. How fast is the plane traveling at that time?
42. A Ferris wheel with a radius of 10 m is rotating at a rate of one revolution every 2 minutes. How fast is a rider rising when his seat is 16 m above ground level?
43. A plane flying with a constant speed of 300 km/h passes over a ground radar station at an altitude of 1 km and climbs at an angle of 30° . At what rate is the distance from the plane to the radar station increasing a minute later?
44. Two people start from the same point. One walks east at 3 mi/h and the other walks northeast at 2 mi/h. How fast is the distance between the people changing after 15 minutes?
45. A runner sprints around a circular track of radius 100 m at a constant speed of 7 m/s. The runner's friend is standing at a distance 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200 m?
46. The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

3.10 Linear Approximations and Differentials

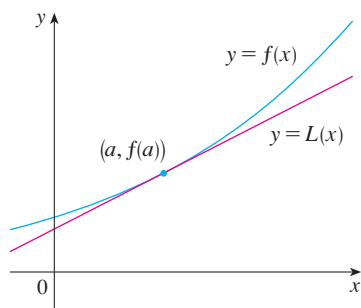


FIGURE 1

We have seen that a curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line. (See Figure 2 in Section 2.7.) This observation is the basis for a method of finding approximate values of functions.

The idea is that it might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f . So we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$. (See Figure 1.)

In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a . An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

and the approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a . The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a .

V EXAMPLE 1 Find the linearization of the function $f(x) = \sqrt{x + 3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

SOLUTION The derivative of $f(x) = (x + 3)^{1/2}$ is

$$f'(x) = \frac{1}{2}(x + 3)^{-1/2} = \frac{1}{2\sqrt{x + 3}}$$

and so we have $f(1) = 2$ and $f'(1) = \frac{1}{4}$. Putting these values into Equation 2, we see that the linearization is

$$L(x) = f(1) + f'(1)(x - 1) = 2 + \frac{1}{4}(x - 1) = \frac{7}{4} + \frac{x}{4}$$

The corresponding linear approximation [1] is

$$\sqrt{x + 3} \approx \frac{7}{4} + \frac{x}{4} \quad (\text{when } x \text{ is near } 1)$$

In particular, we have

$$\sqrt{3.98} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995 \quad \text{and} \quad \sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125$$

The linear approximation is illustrated in Figure 2. We see that, indeed, the tangent line approximation is a good approximation to the given function when x is near 1. We also see that our approximations are overestimates because the tangent line lies above the curve.

Of course, a calculator could give us approximations for $\sqrt{3.98}$ and $\sqrt{4.05}$, but the linear approximation gives an approximation *over an entire interval*.

In the following table we compare the estimates from the linear approximation in Example 1 with the true values. Notice from this table, and also from Figure 2, that the tangent line approximation gives good estimates when x is close to 1 but the accuracy of the approximation deteriorates when x is farther away from 1.

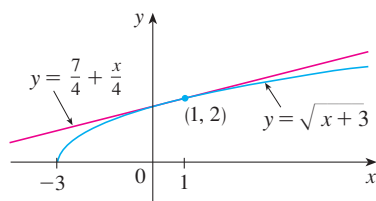


FIGURE 2

	x	From $L(x)$	Actual value
$\sqrt{3.9}$	0.9	1.975	1.97484176 ...
$\sqrt{3.98}$	0.98	1.995	1.99499373 ...
$\sqrt{4}$	1	2	2.00000000 ...
$\sqrt{4.05}$	1.05	2.0125	2.01246117 ...
$\sqrt{4.1}$	1.1	2.025	2.02484567 ...
$\sqrt{5}$	2	2.25	2.23606797 ...
$\sqrt{6}$	3	2.5	2.44948974 ...

How good is the approximation that we obtained in Example 1? The next example shows that by using a graphing calculator or computer we can determine an interval throughout which a linear approximation provides a specified accuracy.

EXAMPLE 2 For what values of x is the linear approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

accurate to within 0.5? What about accuracy to within 0.1?

SOLUTION Accuracy to within 0.5 means that the functions should differ by less than 0.5:

$$\left| \sqrt{x+3} - \left(\frac{7}{4} + \frac{x}{4} \right) \right| < 0.5$$

Equivalently, we could write

$$\sqrt{x+3} - 0.5 < \frac{7}{4} + \frac{x}{4} < \sqrt{x+3} + 0.5$$

This says that the linear approximation should lie between the curves obtained by shifting the curve $y = \sqrt{x+3}$ upward and downward by an amount 0.5. Figure 3 shows the tangent line $y = (7+x)/4$ intersecting the upper curve $y = \sqrt{x+3} + 0.5$ at P and Q . Zooming in and using the cursor, we estimate that the x -coordinate of P is about -2.66 and the x -coordinate of Q is about 8.66 . Thus we see from the graph that the approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

is accurate to within 0.5 when $-2.6 < x < 8.6$. (We have rounded to be safe.)

Similarly, from Figure 4 we see that the approximation is accurate to within 0.1 when $-1.1 < x < 3.9$.

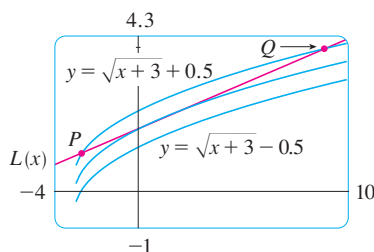


FIGURE 3

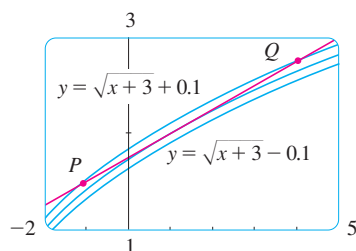


FIGURE 4

Applications to Physics

Linear approximations are often used in physics. In analyzing the consequences of an equation, a physicist sometimes needs to simplify a function by replacing it with its linear approximation. For instance, in deriving a formula for the period of a pendulum, physics textbooks obtain the expression $a_T = -g \sin \theta$ for tangential acceleration and then replace $\sin \theta$ by θ with the remark that $\sin \theta$ is very close to θ if θ is not too large. [See, for example, *Physics: Calculus*, 2d ed., by Eugene Hecht (Pacific Grove, CA, 2000), p. 431.] You can verify that the linearization of the function $f(x) = \sin x$ at $a = 0$ is $L(x) = x$ and so the linear approximation at 0 is

$$\sin x \approx x$$

(see Exercise 42). So, in effect, the derivation of the formula for the period of a pendulum uses the tangent line approximation for the sine function.

Another example occurs in the theory of optics, where light rays that arrive at shallow angles relative to the optical axis are called *paraxial rays*. In paraxial (or Gaussian) optics, both $\sin \theta$ and $\cos \theta$ are replaced by their linearizations. In other words, the linear approximations

$$\sin \theta \approx \theta \quad \text{and} \quad \cos \theta \approx 1$$

are used because θ is close to 0. The results of calculations made with these approximations became the basic theoretical tool used to design lenses. [See *Optics*, 4th ed., by Eugene Hecht (San Francisco, 2002), p. 154.]

In Section 11.11 we will present several other applications of the idea of linear approximations to physics and engineering.

Differentials

The ideas behind linear approximations are sometimes formulated in the terminology and notation of *differentials*. If $y = f(x)$, where f is a differentiable function, then the **differential** dx is an independent variable; that is, dx can be given the value of any real number. The **differential** dy is then defined in terms of dx by the equation

$$\boxed{3} \quad dy = f'(x) dx$$

So dy is a dependent variable; it depends on the values of x and dx . If dx is given a specific value and x is taken to be some specific number in the domain of f , then the numerical value of dy is determined.

The geometric meaning of differentials is shown in Figure 5. Let $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$ be points on the graph of f and let $dx = \Delta x$. The corresponding change in y is

$$\Delta y = f(x + \Delta x) - f(x)$$

The slope of the tangent line PR is the derivative $f'(x)$. Thus the directed distance from S to R is $f'(x) dx = dy$. Therefore dy represents the amount that the tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount dx .

If $dx \neq 0$, we can divide both sides of Equation 3 by dx to obtain

$$\frac{dy}{dx} = f'(x)$$

We have seen similar equations before, but now the left side can genuinely be interpreted as a ratio of differentials.

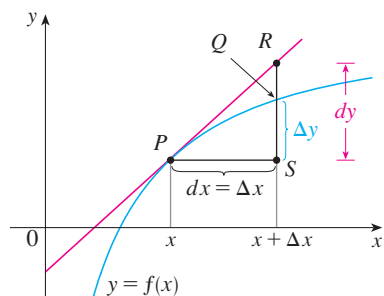


FIGURE 5

EXAMPLE 3 Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes (a) from 2 to 2.05 and (b) from 2 to 2.01.

SOLUTION

(a) We have

$$f(2) = 2^3 + 2^2 - 2(2) + 1 = 9$$

$$f(2.05) = (2.05)^3 + (2.05)^2 - 2(2.05) + 1 = 9.717625$$

$$\Delta y = f(2.05) - f(2) = 0.717625$$

In general,

$$dy = f'(x) dx = (3x^2 + 2x - 2) dx$$

When $x = 2$ and $dx = \Delta x = 0.05$, this becomes

$$dy = [3(2)^2 + 2(2) - 2]0.05 = 0.7$$

$$(b) \quad f(2.01) = (2.01)^3 + (2.01)^2 - 2(2.01) + 1 = 9.140701$$

$$\Delta y = f(2.01) - f(2) = 0.140701$$

When $dx = \Delta x = 0.01$,

$$dy = [3(2)^2 + 2(2) - 2]0.01 = 0.14$$

Figure 6 shows the function in Example 3 and a comparison of dy and Δy when $a = 2$. The viewing rectangle is $[1.8, 2.5]$ by $[6, 18]$.

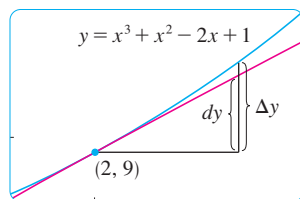


FIGURE 6

Notice that the approximation $\Delta y \approx dy$ becomes better as Δx becomes smaller in Example 3. Notice also that dy was easier to compute than Δy . For more complicated functions it may be impossible to compute Δy exactly. In such cases the approximation by differentials is especially useful.

In the notation of differentials, the linear approximation [1] can be written as

$$f(a + dx) \approx f(a) + dy$$

For instance, for the function $f(x) = \sqrt{x + 3}$ in Example 1, we have

$$dy = f'(x) dx = \frac{dx}{2\sqrt{x + 3}}$$

If $a = 1$ and $dx = \Delta x = 0.05$, then

$$dy = \frac{0.05}{2\sqrt{1 + 3}} = 0.0125$$

and

$$\sqrt{4.05} = f(1.05) \approx f(1) + dy = 2.0125$$

just as we found in Example 1.

Our final example illustrates the use of differentials in estimating the errors that occur because of approximate measurements.

V EXAMPLE 4 The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

SOLUTION If the radius of the sphere is r , then its volume is $V = \frac{4}{3}\pi r^3$. If the error in the measured value of r is denoted by $dr = \Delta r$, then the corresponding error in the calculated value of V is ΔV , which can be approximated by the differential

$$dV = 4\pi r^2 dr$$

When $r = 21$ and $dr = 0.05$, this becomes

$$dV = 4\pi(21)^2 0.05 \approx 277$$

The maximum error in the calculated volume is about 277 cm^3 .

NOTE Although the possible error in Example 4 may appear to be rather large, a better picture of the error is given by the **relative error**, which is computed by dividing the error by the total volume:

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = 3 \frac{dr}{r}$$

Thus the relative error in the volume is about three times the relative error in the radius. In Example 4 the relative error in the radius is approximately $dr/r = 0.05/21 \approx 0.0024$ and it produces a relative error of about 0.007 in the volume. The errors could also be expressed as **percentage errors** of 0.24% in the radius and 0.7% in the volume.

3.10 Exercises


1–4 Find the linearization $L(x)$ of the function at a .


1. $f(x) = x^4 + 3x^2$, $a = -1$


2. $f(x) = \sin x$, $a = \pi/6$

3. $f(x) = \sqrt{x}$, $a = 4$

4. $f(x) = x^{3/4}$, $a = 16$

 5. Find the linear approximation of the function $f(x) = \sqrt{1-x}$ at $a = 0$ and use it to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$. Illustrate by graphing f and the tangent line.

 6. Find the linear approximation of the function $g(x) = \sqrt[3]{1+x}$ at $a = 0$ and use it to approximate the numbers $\sqrt[3]{0.95}$ and $\sqrt[3]{1.1}$. Illustrate by graphing g and the tangent line.

 7–10 Verify the given linear approximation at $a = 0$. Then determine the values of x for which the linear approximation is accurate to within 0.1.

7. $\ln(1+x) \approx x$

8. $(1+x)^{-3} \approx 1-3x$

9. $\sqrt[4]{1+2x} \approx 1 + \frac{1}{2}x$

10. $e^x \cos x \approx 1+x$

11–14 Find the differential of each function.

11. (a) $y = x^2 \sin 2x$

(b) $y = \ln \sqrt{1+t^2}$

12. (a) $y = s/(1+2s)$

(b) $y = e^{-u} \cos u$

13. (a) $y = \tan \sqrt{t}$

(b) $y = \frac{1-v^2}{1+v^2}$

14. (a) $y = e^{\tan \pi t}$

(b) $y = \sqrt{1+\ln z}$

15–18 (a) Find the differential dy and (b) evaluate dy for the given values of x and dx .

15. $y = e^{x/10}$, $x = 0$, $dx = 0.1$

16. $y = \cos \pi x$, $x = \frac{1}{3}$, $dx = -0.02$

17. $y = \sqrt{3+x^2}$, $x = 1$, $dx = -0.1$

18. $y = \frac{x+1}{x-1}$, $x = 2$, $dx = 0.05$

19–22 Compute Δy and dy for the given values of x and $dx = \Delta x$. Then sketch a diagram like Figure 5 showing the line segments with lengths dx , dy , and Δy .

19. $y = 2x - x^2$, $x = 2$, $\Delta x = -0.4$

20. $y = \sqrt{x}$, $x = 1$, $\Delta x = 1$

21. $y = 2/x$, $x = 4$, $\Delta x = 1$

22. $y = e^x$, $x = 0$, $\Delta x = 0.5$

23–28 Use a linear approximation (or differentials) to estimate the given number.

23. $(1.999)^4$

24. $e^{-0.015}$

25. $\sqrt[3]{1001}$

26. $1/4.002$

27. $\tan 44^\circ$

28. $\sqrt{99.8}$

29–31 Explain, in terms of linear approximations or differentials, why the approximation is reasonable.

29. $\sec 0.08 \approx 1$

30. $(1.01)^6 \approx 1.06$

31. $\ln 1.05 \approx 0.05$

32. Let $f(x) = (x-1)^2$ $g(x) = e^{-2x}$

and $h(x) = 1 + \ln(1-2x)$

(a) Find the linearizations of f , g , and h at $a = 0$. What do you notice? How do you explain what happened?(b) Graph f , g , and h and their linear approximations. For which function is the linear approximation best? For which is it worst? Explain.

33. The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error, relative error, and percentage error in computing (a) the volume of the cube and (b) the surface area of the cube.

34. The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm.

(a) Use differentials to estimate the maximum error in the calculated area of the disk.

(b) What is the relative error? What is the percentage error?

35. The circumference of a sphere was measured to be 84 cm with a possible error of 0.5 cm.

(a) Use differentials to estimate the maximum error in the calculated surface area. What is the relative error?

(b) Use differentials to estimate the maximum error in the calculated volume. What is the relative error?

36. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.

37. (a) Use differentials to find a formula for the approximate volume of a thin cylindrical shell with height h , inner radius r , and thickness Δr .

(b) What is the error involved in using the formula from part (a)?

38. One side of a right triangle is known to be 20 cm long and the opposite angle is measured as 30° , with a possible error of $\pm 1^\circ$.

(a) Use differentials to estimate the error in computing the length of the hypotenuse.

(b) What is the percentage error?

39. If a current I passes through a resistor with resistance R , Ohm's Law states that the voltage drop is $V = RI$. If V is constant and R is measured with a certain error, use differentials to show that the relative error in calculating I is approximately the same (in magnitude) as the relative error in R .

40. When blood flows along a blood vessel, the flux F (the volume of blood per unit time that flows past a given point) is proportional to the fourth power of the radius R of the blood vessel:

$$F = kR^4$$

(This is known as Poiseuille's Law; we will show why it is true in Section 8.4.) A partially clogged artery can be expanded by an operation called angioplasty, in which a balloon-tipped catheter is inflated inside the artery in order to widen it and restore the normal blood flow.

Show that the relative change in F is about four times the relative change in R . How will a 5% increase in the radius affect the flow of blood?

41. Establish the following rules for working with differentials (where c denotes a constant and u and v are functions of x).

(a) $dc = 0$	(b) $d(cu) = c \, du$
(c) $d(u + v) = du + dv$	(d) $d(uv) = u \, dv + v \, du$
(e) $d\left(\frac{u}{v}\right) = \frac{v \, du - u \, dv}{v^2}$	(f) $d(x^n) = nx^{n-1} dx$

42. On page 431 of *Physics: Calculus*, 2d ed., by Eugene Hecht (Pacific Grove, CA, 2000), in the course of deriving the formula $T = 2\pi\sqrt{L/g}$ for the period of a pendulum of length L , the author obtains the equation $a_T = -g \sin \theta$ for the tangen-

tial acceleration of the bob of the pendulum. He then says, "for small angles, the value of θ in radians is very nearly the value of $\sin \theta$; they differ by less than 2% out to about 20° ."

- (a) Verify the linear approximation at 0 for the sine function:

$$\sin x \approx x$$

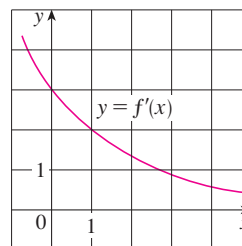
- (b) Use a graphing device to determine the values of x for which $\sin x$ and x differ by less than 2%. Then verify Hecht's statement by converting from radians to degrees.

43. Suppose that the only information we have about a function f is that $f(1) = 5$ and the graph of its derivative is as shown.

- (a) Use a linear approximation to estimate $f(0.9)$ and $f(1.1)$.

- (b) Are your estimates in part (a) too large or too small?

Explain.



44. Suppose that we don't have a formula for $g(x)$ but we know that $g(2) = -4$ and $g'(x) = \sqrt{x^2 + 5}$ for all x .

- (a) Use a linear approximation to estimate $g(1.95)$ and $g(2.05)$.

- (b) Are your estimates in part (a) too large or too small?

Explain.

LABORATORY PROJECT TAYLOR POLYNOMIALS

The tangent line approximation $L(x)$ is the best first-degree (linear) approximation to $f(x)$ near $x = a$ because $f(x)$ and $L(x)$ have the same rate of change (derivative) at a . For a better approximation than a linear one, let's try a second-degree (quadratic) approximation $P(x)$. In other words, we approximate a curve by a parabola instead of by a straight line. To make sure that the approximation is a good one, we stipulate the following:

- | | |
|-------------------------|---|
| (i) $P(a) = f(a)$ | (P and f should have the same value at a .) |
| (ii) $P'(a) = f'(a)$ | (P and f should have the same rate of change at a .) |
| (iii) $P''(a) = f''(a)$ | (The slopes of P and f should change at the same rate at a .) |

- Find the quadratic approximation $P(x) = A + Bx + Cx^2$ to the function $f(x) = \cos x$ that satisfies conditions (i), (ii), and (iii) with $a = 0$. Graph P , f , and the linear approximation $L(x) = 1$ on a common screen. Comment on how well the functions P and L approximate f .
- Determine the values of x for which the quadratic approximation $f(x) \approx P(x)$ in Problem 1 is accurate to within 0.1. [Hint: Graph $y = P(x)$, $y = \cos x - 0.1$, and $y = \cos x + 0.1$ on a common screen.]



Graphing calculator or computer required

3. To approximate a function f by a quadratic function P near a number a , it is best to write P in the form

$$P(x) = A + B(x - a) + C(x - a)^2$$

Show that the quadratic function that satisfies conditions (i), (ii), and (iii) is

$$P(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$$

4. Find the quadratic approximation to $f(x) = \sqrt{x+3}$ near $a = 1$. Graph f , the quadratic approximation, and the linear approximation from Example 2 in Section 3.10 on a common screen. What do you conclude?
5. Instead of being satisfied with a linear or quadratic approximation to $f(x)$ near $x = a$, let's try to find better approximations with higher-degree polynomials. We look for an n th-degree polynomial

$$T_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots + c_n(x - a)^n$$

such that T_n and its first n derivatives have the same values at $x = a$ as f and its first n derivatives. By differentiating repeatedly and setting $x = a$, show that these conditions are satisfied if $c_0 = f(a)$, $c_1 = f'(a)$, $c_2 = \frac{1}{2}f''(a)$, and in general

$$c_k = \frac{f^{(k)}(a)}{k!}$$

where $k! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot k$. The resulting polynomial

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

is called the **n th-degree Taylor polynomial of f centered at a** .

6. Find the 8th-degree Taylor polynomial centered at $a = 0$ for the function $f(x) = \cos x$. Graph f together with the Taylor polynomials T_2, T_4, T_6, T_8 in the viewing rectangle $[-5, 5]$ by $[-1.4, 1.4]$ and comment on how well they approximate f .

3.11 Hyperbolic Functions

Certain even and odd combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics and its applications that they deserve to be given special names. In many ways they are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason they are collectively called **hyperbolic functions** and individually called **hyperbolic sine**, **hyperbolic cosine**, and so on.

Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{coth} x = \frac{\cosh x}{\sinh x}$$

The graphs of hyperbolic sine and cosine can be sketched using graphical addition as in Figures 1 and 2.

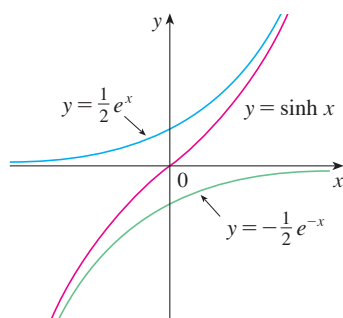


FIGURE 1

$$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$$

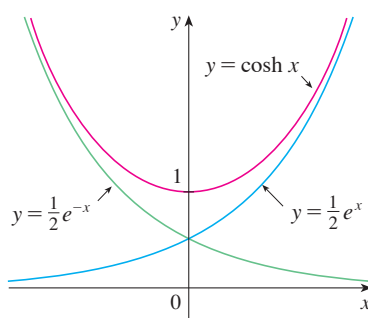


FIGURE 2

$$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$$

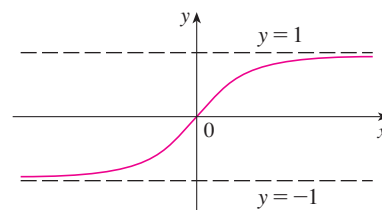


FIGURE 3

$$y = \tanh x$$

Note that \sinh has domain \mathbb{R} and range \mathbb{R} , while \cosh has domain \mathbb{R} and range $[1, \infty)$. The graph of \tanh is shown in Figure 3. It has the horizontal asymptotes $y = \pm 1$. (See Exercise 23.)

Some of the mathematical uses of hyperbolic functions will be seen in Chapter 7. Applications to science and engineering occur whenever an entity such as light, velocity, electricity, or radioactivity is gradually absorbed or extinguished, for the decay can be represented by hyperbolic functions. The most famous application is the use of hyperbolic cosine to describe the shape of a hanging wire. It can be proved that if a heavy flexible cable (such as a telephone or power line) is suspended between two points at the same height, then it takes the shape of a curve with equation $y = c + a \cosh(x/a)$ called a *catenary* (see Figure 4). (The Latin word *catena* means “chain.”)

Another application of hyperbolic functions occurs in the description of ocean waves: The velocity of a water wave with length L moving across a body of water with depth d is modeled by the function

$$v = \sqrt{\frac{gL}{2\pi} \tanh\left(\frac{2\pi d}{L}\right)}$$

where g is the acceleration due to gravity. (See Figure 5 and Exercise 49.)

The hyperbolic functions satisfy a number of identities that are similar to well-known trigonometric identities. We list some of them here and leave most of the proofs to the exercises.

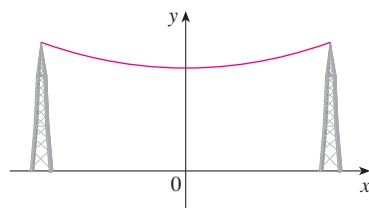


FIGURE 4

A catenary $y = c + a \cosh(x/a)$

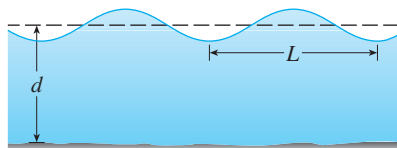


FIGURE 5

Idealized ocean wave

Hyperbolic Identities

$$\sinh(-x) = -\sinh x$$

$$\cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$



© 2006 Getty Images

The Gateway Arch in St. Louis was designed using a hyperbolic cosine function (Exercise 48).

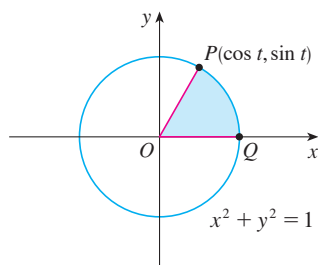


FIGURE 6

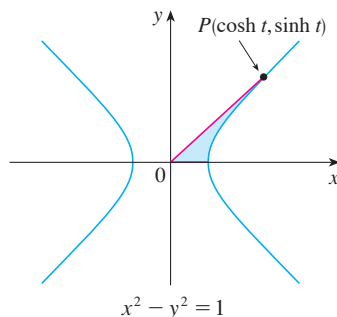


FIGURE 7

V EXAMPLE 1 Prove (a) $\cosh^2 x - \sinh^2 x = 1$ and (b) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

SOLUTION

$$\begin{aligned} \text{(a)} \quad \cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2} \right)^2 - \left(\frac{e^x - e^{-x}}{2} \right)^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{4}{4} = 1 \end{aligned}$$

(b) We start with the identity proved in part (a):

$$\cosh^2 x - \sinh^2 x = 1$$

If we divide both sides by $\cosh^2 x$, we get

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

or

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

The identity proved in Example 1(a) gives a clue to the reason for the name “hyperbolic” functions:

If t is any real number, then the point $P(\cos t, \sin t)$ lies on the unit circle $x^2 + y^2 = 1$ because $\cos^2 t + \sin^2 t = 1$. In fact, t can be interpreted as the radian measure of $\angle POQ$ in Figure 6. For this reason the trigonometric functions are sometimes called *circular* functions.

Likewise, if t is any real number, then the point $P(\cosh t, \sinh t)$ lies on the right branch of the hyperbola $x^2 - y^2 = 1$ because $\cosh^2 t - \sinh^2 t = 1$ and $\cosh t \geq 1$. This time, t does not represent the measure of an angle. However, it turns out that t represents twice the area of the shaded hyperbolic sector in Figure 7, just as in the trigonometric case t represents twice the area of the shaded circular sector in Figure 6.

The derivatives of the hyperbolic functions are easily computed. For example,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

We list the differentiation formulas for the hyperbolic functions as Table 1. The remaining proofs are left as exercises. Note the analogy with the differentiation formulas for trigonometric functions, but beware that the signs are different in some cases.

1 Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

EXAMPLE 2 Any of these differentiation rules can be combined with the Chain Rule. For instance,

$$\frac{d}{dx}(\cosh \sqrt{x}) = \sinh \sqrt{x} \cdot \frac{d}{dx} \sqrt{x} = \frac{\sinh \sqrt{x}}{2\sqrt{x}}$$

Inverse Hyperbolic Functions

You can see from Figures 1 and 3 that \sinh and \tanh are one-to-one functions and so they have inverse functions denoted by \sinh^{-1} and \tanh^{-1} . Figure 2 shows that \cosh is not one-to-one, but when restricted to the domain $[0, \infty)$ it becomes one-to-one. The inverse hyperbolic cosine function is defined as the inverse of this restricted function.

2

$$y = \sinh^{-1}x \iff \sinh y = x$$

$$y = \cosh^{-1}x \iff \cosh y = x \quad \text{and} \quad y \geq 0$$

$$y = \tanh^{-1}x \iff \tanh y = x$$

The remaining inverse hyperbolic functions are defined similarly (see Exercise 28).

We can sketch the graphs of \sinh^{-1} , \cosh^{-1} , and \tanh^{-1} in Figures 8, 9, and 10 by using Figures 1, 2, and 3.

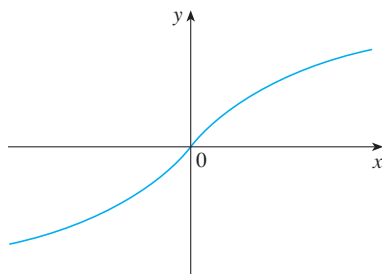


FIGURE 8 $y = \sinh^{-1}x$
domain = \mathbb{R} range = \mathbb{R}

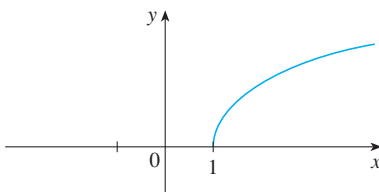


FIGURE 9 $y = \cosh^{-1}x$
domain = $[1, \infty)$ range = $[0, \infty)$

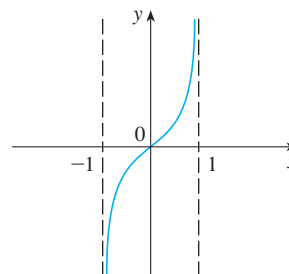


FIGURE 10 $y = \tanh^{-1}x$
domain = $(-1, 1)$ range = \mathbb{R}

Since the hyperbolic functions are defined in terms of exponential functions, it's not surprising to learn that the inverse hyperbolic functions can be expressed in terms of logarithms. In particular, we have:

3

$$\sinh^{-1}x = \ln(x + \sqrt{x^2 + 1}) \quad x \in \mathbb{R}$$

4

$$\cosh^{-1}x = \ln(x + \sqrt{x^2 - 1}) \quad x \geq 1$$

5

$$\tanh^{-1}x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \quad -1 < x < 1$$

Formula 3 is proved in Example 3. The proofs of Formulas 4 and 5 are requested in Exercises 26 and 27.

EXAMPLE 3 Show that $\sinh^{-1}x = \ln(x + \sqrt{x^2 + 1})$.

SOLUTION Let $y = \sinh^{-1}x$. Then

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

so
$$e^y - 2x - e^{-y} = 0$$

or, multiplying by e^y ,

$$e^{2y} - 2xe^y - 1 = 0$$

This is really a quadratic equation in e^y :

$$(e^y)^2 - 2x(e^y) - 1 = 0$$

Solving by the quadratic formula, we get

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$

Note that $e^y > 0$, but $x - \sqrt{x^2 + 1} < 0$ (because $x < \sqrt{x^2 + 1}$). Thus the minus sign is inadmissible and we have

$$e^y = x + \sqrt{x^2 + 1}$$

Therefore
$$y = \ln(e^y) = \ln(x + \sqrt{x^2 + 1})$$

(See Exercise 25 for another method.)

6 Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx} (\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx} (\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2+1}}$$

$$\frac{d}{dx} (\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx} (\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx} (\tanh^{-1}x) = \frac{1}{1-x^2}$$

$$\frac{d}{dx} (\operatorname{coth}^{-1}x) = \frac{1}{1-x^2}$$

Notice that the formulas for the derivatives of $\tanh^{-1}x$ and $\operatorname{coth}^{-1}x$ appear to be identical. But the domains of these functions have no numbers in common: $\tanh^{-1}x$ is defined for $|x| < 1$, whereas $\operatorname{coth}^{-1}x$ is defined for $|x| > 1$.

The inverse hyperbolic functions are all differentiable because the hyperbolic functions are differentiable. The formulas in Table 6 can be proved either by the method for inverse functions or by differentiating Formulas 3, 4, and 5.

V EXAMPLE 4 Prove that $\frac{d}{dx} (\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$.

SOLUTION 1 Let $y = \sinh^{-1}x$. Then $\sinh y = x$. If we differentiate this equation implicitly with respect to x , we get

$$\cosh y \frac{dy}{dx} = 1$$

Since $\cosh^2 y - \sinh^2 y = 1$ and $\cosh y \geq 0$, we have $\cosh y = \sqrt{1 + \sinh^2 y}$, so

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}$$

SOLUTION 2 From Equation 3 (proved in Example 3), we have

$$\begin{aligned}
 \frac{d}{dx}(\sinh^{-1}x) &= \frac{d}{dx} \ln(x + \sqrt{x^2 + 1}) \\
 &= \frac{1}{x + \sqrt{x^2 + 1}} \frac{d}{dx}(x + \sqrt{x^2 + 1}) \\
 &= \frac{1}{x + \sqrt{x^2 + 1}} \left(1 + \frac{x}{\sqrt{x^2 + 1}}\right) \\
 &= \frac{\sqrt{x^2 + 1} + x}{(x + \sqrt{x^2 + 1})\sqrt{x^2 + 1}} \\
 &= \frac{1}{\sqrt{x^2 + 1}}
 \end{aligned}$$

V EXAMPLE 5 Find $\frac{d}{dx}[\tanh^{-1}(\sin x)]$.

SOLUTION Using Table 6 and the Chain Rule, we have

$$\begin{aligned}
 \frac{d}{dx}[\tanh^{-1}(\sin x)] &= \frac{1}{1 - (\sin x)^2} \frac{d}{dx}(\sin x) \\
 &= \frac{1}{1 - \sin^2 x} \cos x = \frac{\cos x}{\cos^2 x} = \sec x
 \end{aligned}$$

3.11 Exercises

1–6 Find the numerical value of each expression.

- | | |
|--------------------------------|--------------------|
| 1. (a) $\sinh 0$ | (b) $\cosh 0$ |
| 2. (a) $\tanh 0$ | (b) $\tanh 1$ |
| 3. (a) $\sinh(\ln 2)$ | (b) $\sinh 2$ |
| 4. (a) $\cosh 3$ | (b) $\cosh(\ln 3)$ |
| 5. (a) $\operatorname{sech} 0$ | (b) $\cosh^{-1} 1$ |
| 6. (a) $\sinh 1$ | (b) $\sinh^{-1} 1$ |

7–19 Prove the identity.

7. $\sinh(-x) = -\sinh x$
(This shows that \sinh is an odd function.)
8. $\cosh(-x) = \cosh x$
(This shows that \cosh is an even function.)
9. $\cosh x + \sinh x = e^x$
10. $\cosh x - \sinh x = e^{-x}$
11. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
12. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

13. $\coth^2 x - 1 = \operatorname{csch}^2 x$

14. $\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

15. $\sinh 2x = 2 \sinh x \cosh x$

16. $\cosh 2x = \cosh^2 x + \sinh^2 x$

17. $\tanh(\ln x) = \frac{x^2 - 1}{x^2 + 1}$

18. $\frac{1 + \tanh x}{1 - \tanh x} = e^{2x}$

19. $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$
(n any real number)

20. If $\tanh x = \frac{12}{13}$, find the values of the other hyperbolic functions at x .

21. If $\cosh x = \frac{5}{3}$ and $x > 0$, find the values of the other hyperbolic functions at x .

22. (a) Use the graphs of \sinh , \cosh , and \tanh in Figures 1–3 to draw the graphs of csch , sech , and \coth .



Graphing calculator or computer required

1. Homework Hints available at stewartcalculus.com



(b) Check the graphs that you sketched in part (a) by using a graphing device to produce them.

23. Use the definitions of the hyperbolic functions to find each of the following limits.

(a) $\lim_{x \rightarrow -\infty} \tanh x$	(b) $\lim_{x \rightarrow -\infty} \tanh x$
(c) $\lim_{x \rightarrow -\infty} \sinh x$	(d) $\lim_{x \rightarrow -\infty} \sinh x$
(e) $\lim_{x \rightarrow -\infty} \operatorname{sech} x$	(f) $\lim_{x \rightarrow -\infty} \coth x$
(g) $\lim_{x \rightarrow 0^+} \coth x$	(h) $\lim_{x \rightarrow 0^-} \coth x$
(i) $\lim_{x \rightarrow -\infty} \operatorname{csch} x$	

24. Prove the formulas given in Table 1 for the derivatives of the functions (a) \cosh , (b) \tanh , (c) csch , (d) sech , and (e) \coth .

25. Give an alternative solution to Example 3 by letting $y = \sinh^{-1}x$ and then using Exercise 9 and Example 1(a) with x replaced by y .

26. Prove Equation 4.

27. Prove Equation 5 using (a) the method of Example 3 and (b) Exercise 18 with x replaced by y .

28. For each of the following functions (i) give a definition like those in [2], (ii) sketch the graph, and (iii) find a formula similar to Equation 3.

(a) csch^{-1} (b) sech^{-1} (c) coth^{-1}

29. Prove the formulas given in Table 6 for the derivatives of the following functions.

(a) \cosh^{-1} (b) \tanh^{-1} (c) csch^{-1}
 (d) sech^{-1} (e) coth^{-1}

- 30–45 Find the derivative. Simplify where possible.

30. $f(x) = \tanh(1 + e^{2x})$ 31. $f(x) = x \sinh x - \cosh x$

32. $g(x) = \cosh(\ln x)$ 33. $h(x) = \ln(\cosh x)$

34. $y = x \coth(1 + x^2)$ 35. $y = e^{\cosh 3x}$

36. $f(t) = \operatorname{csch} t(1 - \ln \operatorname{csch} t)$ 37. $f(t) = \operatorname{sech}^2(e^t)$

38. $y = \sinh(\cosh x)$ 39. $G(x) = \frac{1 - \cosh x}{1 + \cosh x}$

40. $y = \sinh^{-1}(\tan x)$ 41. $y = \cosh^{-1}\sqrt{x}$

42. $y = x \tanh^{-1}x + \ln \sqrt{1 - x^2}$

43. $y = x \sinh^{-1}(x/3) - \sqrt{9 + x^2}$

44. $y = \operatorname{sech}^{-1}(e^{-x})$

45. $y = \operatorname{coth}^{-1}(\sec x)$

46. Show that $\frac{d}{dx} \sqrt[4]{\frac{1 + \tanh x}{1 - \tanh x}} = \frac{1}{2}e^{x/2}$

47. Show that $\frac{d}{dx} \arctan(\tanh x) = \operatorname{sech} 2x$.

48. The Gateway Arch in St. Louis was designed by Eero Saarinen and was constructed using the equation

$$y = 211.49 - 20.96 \cosh 0.03291765x$$

for the central curve of the arch, where x and y are measured in meters and $|x| \leq 91.20$.



- (a) Graph the central curve.
 (b) What is the height of the arch at its center?
 (c) At what points is the height 100 m?
 (d) What is the slope of the arch at the points in part (c)?

49. If a water wave with length L moves with velocity v in a body of water with depth d , then

$$v = \sqrt{\frac{gL}{2\pi} \tanh\left(\frac{2\pi d}{L}\right)}$$

where g is the acceleration due to gravity. (See Figure 5.) Explain why the approximation

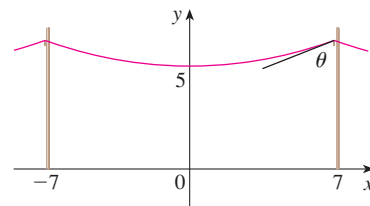
$$v \approx \sqrt{\frac{gL}{2\pi}}$$

is appropriate in deep water.



50. A flexible cable always hangs in the shape of a catenary $y = c + a \cosh(x/a)$, where c and a are constants and $a > 0$ (see Figure 4 and Exercise 52). Graph several members of the family of functions $y = a \cosh(x/a)$. How does the graph change as a varies?

51. A telephone line hangs between two poles 14 m apart in the shape of the catenary $y = 20 \cosh(x/20) - 15$, where x and y are measured in meters.
 (a) Find the slope of this curve where it meets the right pole.
 (b) Find the angle θ between the line and the pole.



52. Using principles from physics it can be shown that when a cable is hung between two poles, it takes the shape of a curve $y = f(x)$ that satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{\rho g}{T} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

where ρ is the linear density of the cable, g is the acceleration due to gravity, T is the tension in the cable at its lowest point, and the coordinate system is chosen appropriately. Verify that the function

$$y = f(x) = \frac{T}{\rho g} \cosh\left(\frac{\rho g x}{T}\right)$$

is a solution of this differential equation.

53. A cable with linear density $\rho = 2$ kg/m is strung from the tops of two poles that are 200 m apart.

- Use Exercise 52 to find the tension T so that the cable is 60 m above the ground at its lowest point. How tall are the poles?
- If the tension is doubled, what is the new low point of the cable? How tall are the poles now?

54. Evaluate $\lim_{x \rightarrow \infty} \frac{\sinh x}{e^x}$.

55. (a) Show that any function of the form

$$y = A \sinh mx + B \cosh mx$$

satisfies the differential equation $y'' = m^2 y$.

- Find $y = y(x)$ such that $y'' = 9y$, $y(0) = -4$, and $y'(0) = 6$.

56. If $x = \ln(\sec \theta + \tan \theta)$, show that $\sec \theta = \cosh x$.

57. At what point of the curve $y = \cosh x$ does the tangent have slope 1?



58. Investigate the family of functions

$$f_n(x) = \tanh(n \sin x)$$

where n is a positive integer. Describe what happens to the graph of f_n when n becomes large.

59. Show that if $a \neq 0$ and $b \neq 0$, then there exist numbers α and β such that $ae^x + be^{-x}$ equals either $\alpha \sinh(x + \beta)$ or $\alpha \cosh(x + \beta)$. In other words, almost every function of the form $f(x) = ae^x + be^{-x}$ is a shifted and stretched hyperbolic sine or cosine function.

3 Review

Concept Check

- State each differentiation rule both in symbols and in words.
 - The Power Rule
 - The Constant Multiple Rule
 - The Sum Rule
 - The Difference Rule
 - The Product Rule
 - The Quotient Rule
 - The Chain Rule

2. State the derivative of each function.

- | | | |
|------------------------|------------------------|------------------------|
| (a) $y = x^n$ | (b) $y = e^x$ | (c) $y = a^x$ |
| (d) $y = \ln x$ | (e) $y = \log_a x$ | (f) $y = \sin x$ |
| (g) $y = \cos x$ | (h) $y = \tan x$ | (i) $y = \csc x$ |
| (j) $y = \sec x$ | (k) $y = \cot x$ | (l) $y = \sin^{-1} x$ |
| (m) $y = \cos^{-1} x$ | (n) $y = \tan^{-1} x$ | (o) $y = \sinh x$ |
| (p) $y = \cosh x$ | (q) $y = \tanh x$ | (r) $y = \sinh^{-1} x$ |
| (s) $y = \cosh^{-1} x$ | (t) $y = \tanh^{-1} x$ | |

- How is the number e defined?
 - Express e as a limit.
 - Why is the natural exponential function $y = e^x$ used more often in calculus than the other exponential functions $y = a^x$?

(d) Why is the natural logarithmic function $y = \ln x$ used more often in calculus than the other logarithmic functions $y = \log_a x$?

- Explain how implicit differentiation works.
 - Explain how logarithmic differentiation works.
- Give several examples of how the derivative can be interpreted as a rate of change in physics, chemistry, biology, economics, or other sciences.
- Write a differential equation that expresses the law of natural growth.
 - Under what circumstances is this an appropriate model for population growth?
 - What are the solutions of this equation?
- Write an expression for the linearization of f at a .
 - If $y = f(x)$, write an expression for the differential dy .
 - If $dx = \Delta x$, draw a picture showing the geometric meanings of Δy and dy .

True-False Quiz

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

- If f and g are differentiable, then

$$\frac{d}{dx} [f(x) + g(x)] = f'(x) + g'(x)$$

- If f and g are differentiable, then

$$\frac{d}{dx} [f(x)g(x)] = f'(x)g'(x)$$

- If f and g are differentiable, then

$$\frac{d}{dx} [f(g(x))] = f'(g(x))g'(x)$$

- If f is differentiable, then $\frac{d}{dx} \sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$.

- If f is differentiable, then $\frac{d}{dx} f(\sqrt{x}) = \frac{f'(x)}{2\sqrt{x}}$.

- If $y = e^2$, then $y' = 2e$.

7. $\frac{d}{dx}(10^x) = x10^{x-1}$

8. $\frac{d}{dx}(\ln 10) = \frac{1}{10}$

9. $\frac{d}{dx}(\tan^2 x) = \frac{d}{dx}(\sec^2 x)$

10. $\frac{d}{dx}|x^2 + x| = |2x + 1|$

11. The derivative of a polynomial is a polynomial.

12. If $f(x) = (x^6 - x^4)^5$, then $f^{(31)}(x) = 0$.

13. The derivative of a rational function is a rational function.

14. An equation of the tangent line to the parabola $y = x^2$ at $(-2, 4)$ is $y - 4 = 2x(x + 2)$.

15. If $g(x) = x^5$, then $\lim_{x \rightarrow 2} \frac{g(x) - g(2)}{x - 2} = 80$

Exercises1–50 Calculate y' .

1. $y = (x^2 + x^3)^4$

2. $y = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt[5]{x^3}}$

3. $y = \frac{x^2 - x + 2}{\sqrt{x}}$

4. $y = \frac{\tan x}{1 + \cos x}$

5. $y = x^2 \sin \pi x$

6. $y = x \cos^{-1} x$

7. $y = \frac{t^4 - 1}{t^4 + 1}$

8. $xe^y = y \sin x$

9. $y = \ln(x \ln x)$

10. $y = e^{mx} \cos nx$

11. $y = \sqrt{x} \cos \sqrt{x}$

12. $y = (\arcsin 2x)^2$

13. $y = \frac{e^{1/x}}{x^2}$

14. $y = \ln \sec x$

15. $y + x \cos y = x^2 y$

16. $y = \left(\frac{u - 1}{u^2 + u + 1} \right)^4$

17. $y = \sqrt{\arctan x}$

18. $y = \cot(\csc x)$

19. $y = \tan\left(\frac{t}{1 + t^2}\right)$

20. $y = e^{x \sec x}$

21. $y = 3^{x \ln x}$

22. $y = \sec(1 + x^2)$

23. $y = (1 - x^{-1})^{-1}$

24. $y = 1/\sqrt[3]{x + \sqrt{x}}$

25. $\sin(xy) = x^2 - y$

26. $y = \sqrt{\sin \sqrt{x}}$

27. $y = \log_5(1 + 2x)$

28. $y = (\cos x)^x$

29. $y = \ln \sin x - \frac{1}{2} \sin^2 x$

30. $y = \frac{(x^2 + 1)^4}{(2x + 1)^3(3x - 1)^5}$

31. $y = x \tan^{-1}(4x)$

32. $y = e^{\cos x} + \cos(e^x)$

33. $y = \ln |\sec 5x + \tan 5x|$

34. $y = 10^{\tan \pi \theta}$

35. $y = \cot(3x^2 + 5)$

36. $y = \sqrt{t \ln(t^4)}$

37. $y = \sin(\tan \sqrt{1 + x^3})$

38. $y = \arctan(\arcsin \sqrt{x})$

39. $y = \tan^2(\sin \theta)$

40. $xe^y = y - 1$

41. $y = \frac{\sqrt{x+1}(2-x)^5}{(x+3)^7}$

42. $y = \frac{(x+\lambda)^4}{x^4 + \lambda^4}$

43. $y = x \sinh(x^2)$

44. $y = \frac{\sin mx}{x}$

45. $y = \ln(\cosh 3x)$

46. $y = \ln \left| \frac{x^2 - 4}{2x + 5} \right|$

47. $y = \cosh^{-1}(\sinh x)$

48. $y = x \tanh^{-1} \sqrt{x}$

49. $y = \cos(e^{\sqrt{\tan 3x}})$

50. $y = \sin^2(\cos \sqrt{\sin \pi x})$

51. If $f(t) = \sqrt{4t + 1}$, find $f''(2)$.

52. If $g(\theta) = \theta \sin \theta$, find $g''(\pi/6)$.

53. Find y'' if $x^6 + y^6 = 1$.

54. Find $f^{(n)}(x)$ if $f(x) = 1/(2 - x)$.

55. Use mathematical induction (page 76) to show that if $f(x) = xe^x$, then $f^{(n)}(x) = (x + n)e^x$.

56. Evaluate $\lim_{t \rightarrow 0} \frac{t^3}{\tan^3(2t)}$.

57–59 Find an equation of the tangent to the curve at the given point.

57. $y = 4 \sin^2 x$, $(\pi/6, 1)$

58. $y = \frac{x^2 - 1}{x^2 + 1}$, $(0, -1)$

59. $y = \sqrt{1 + 4 \sin x}$, $(0, 1)$

60–61 Find equations of the tangent line and normal line to the curve at the given point.

60. $x^2 + 4xy + y^2 = 13$, $(2, 1)$

61. $y = (2 + x)e^{-x}$, $(0, 2)$


62. If $f(x) = xe^{\sin x}$, find $f'(x)$. Graph f and f' on the same screen and comment.

63. (a) If $f(x) = x\sqrt{5 - x}$, find $f'(x)$.

(b) Find equations of the tangent lines to the curve $y = x\sqrt{5 - x}$ at the points $(1, 2)$ and $(4, 4)$.

(c) Illustrate part (b) by graphing the curve and tangent lines on the same screen.

(d) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f' .

-  64. (a) If $f(x) = 4x - \tan x$, $-\pi/2 < x < \pi/2$, find f' and f'' .
 (b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f , f' , and f'' .

65. At what points on the curve $y = \sin x + \cos x$, $0 \leq x \leq 2\pi$, is the tangent line horizontal?

66. Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

67. If $f(x) = (x - a)(x - b)(x - c)$, show that

$$\frac{f'(x)}{f(x)} = \frac{1}{x - a} + \frac{1}{x - b} + \frac{1}{x - c}$$

68. (a) By differentiating the double-angle formula

$$\cos 2x = \cos^2 x - \sin^2 x$$

obtain the double-angle formula for the sine function.

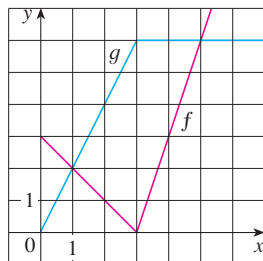
(b) By differentiating the addition formula

$$\sin(x + a) = \sin x \cos a + \cos x \sin a$$

obtain the addition formula for the cosine function.

69. Suppose that $h(x) = f(x)g(x)$ and $F(x) = f(g(x))$, where $f(2) = 3$, $g(2) = 5$, $g'(2) = 4$, $f'(2) = -2$, and $f'(5) = 11$. Find (a) $h'(2)$ and (b) $F'(2)$.

70. If f and g are the functions whose graphs are shown, let $P(x) = f(x)g(x)$, $Q(x) = f(x)/g(x)$, and $C(x) = f(g(x))$. Find (a) $P'(2)$, (b) $Q'(2)$, and (c) $C'(2)$.



71–78 Find f' in terms of g' .

71. $f(x) = x^2 g(x)$

72. $f(x) = g(x^2)$

73. $f(x) = [g(x)]^2$

74. $f(x) = g(g(x))$

75. $f(x) = g(e^x)$

76. $f(x) = e^{g(x)}$

77. $f(x) = \ln |g(x)|$


78. $f(x) = g(\ln x)$

79–81 Find h' in terms of f' and g' .

79. $h(x) = \frac{f(x)g(x)}{f(x) + g(x)}$

80. $h(x) = \sqrt{\frac{f(x)}{g(x)}}$

81. $h(x) = f(g(\sin 4x))$

-  82. (a) Graph the function $f(x) = x - 2 \sin x$ in the viewing rectangle $[0, 8]$ by $[-2, 8]$.
 (b) On which interval is the average rate of change larger: $[1, 2]$ or $[2, 3]$?
 (c) At which value of x is the instantaneous rate of change larger: $x = 2$ or $x = 5$?
 (d) Check your visual estimates in part (c) by computing $f'(x)$ and comparing the numerical values of $f'(2)$ and $f'(5)$.

83. At what point on the curve $y = [\ln(x + 4)]^2$ is the tangent horizontal?

84. (a) Find an equation of the tangent to the curve $y = e^x$ that is parallel to the line $x - 4y = 1$.
 (b) Find an equation of the tangent to the curve $y = e^x$ that passes through the origin.

85. Find a parabola $y = ax^2 + bx + c$ that passes through the point $(1, 4)$ and whose tangent lines at $x = -1$ and $x = 5$ have slopes 6 and -2 , respectively.

86. The function $C(t) = K(e^{-at} - e^{-bt})$, where a , b , and K are positive constants and $b > a$, is used to model the concentration at time t of a drug injected into the bloodstream.

(a) Show that $\lim_{t \rightarrow \infty} C(t) = 0$.

(b) Find $C'(t)$, the rate at which the drug is cleared from circulation.

(c) When is this rate equal to 0?

87. An equation of motion of the form $s = Ae^{-ct} \cos(\omega t + \delta)$ represents damped oscillation of an object. Find the velocity and acceleration of the object.

88. A particle moves along a horizontal line so that its coordinate at time t is $x = \sqrt{b^2 + c^2 t^2}$, $t \geq 0$, where b and c are positive constants.

(a) Find the velocity and acceleration functions.


(b) Show that the particle always moves in the positive direction.

89. A particle moves on a vertical line so that its coordinate at time t is $y = t^3 - 12t + 3$, $t \geq 0$.

(a) Find the velocity and acceleration functions.

(b) When is the particle moving upward and when is it moving downward?

(c) Find the distance that the particle travels in the time interval $0 \leq t \leq 3$.

 (d) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 3$.

(e) When is the particle speeding up? When is it slowing down?

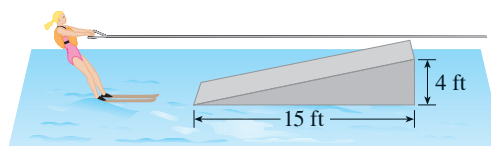
90. The volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.

(a) Find the rate of change of the volume with respect to the height if the radius is constant.

- (b) Find the rate of change of the volume with respect to the radius if the height is constant.

91. The mass of part of a wire is $x(1 + \sqrt{x})$ kilograms, where x is measured in meters from one end of the wire. Find the linear density of the wire when $x = 4$ m.
92. The cost, in dollars, of producing x units of a certain commodity is
- $$C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3$$
- (a) Find the marginal cost function.
 (b) Find $C'(100)$ and explain its meaning.
 (c) Compare $C'(100)$ with the cost of producing the 101st item.
93. A bacteria culture contains 200 cells initially and grows at a rate proportional to its size. After half an hour the population has increased to 360 cells.
 (a) Find the number of bacteria after t hours.
 (b) Find the number of bacteria after 4 hours.
 (c) Find the rate of growth after 4 hours.
 (d) When will the population reach 10,000?
94. Cobalt-60 has a half-life of 5.24 years.
 (a) Find the mass that remains from a 100-mg sample after 20 years.
 (b) How long would it take for the mass to decay to 1 mg?
95. Let $C(t)$ be the concentration of a drug in the bloodstream. As the body eliminates the drug, $C(t)$ decreases at a rate that is proportional to the amount of the drug that is present at the time. Thus $C'(t) = -kC(t)$, where k is a positive number called the *elimination constant* of the drug.
 (a) If C_0 is the concentration at time $t = 0$, find the concentration at time t .
 (b) If the body eliminates half the drug in 30 hours, how long does it take to eliminate 90% of the drug?
96. A cup of hot chocolate has temperature 80°C in a room kept at 20°C . After half an hour the hot chocolate cools to 60°C .
 (a) What is the temperature of the chocolate after another half hour?
 (b) When will the chocolate have cooled to 40°C ?
97. The volume of a cube is increasing at a rate of $10 \text{ cm}^3/\text{min}$. How fast is the surface area increasing when the length of an edge is 30 cm?
98. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of $2 \text{ cm}^3/\text{s}$, how fast is the water level rising when the water is 5 cm deep?
99. A balloon is rising at a constant speed of 5 ft/s. A boy is cycling along a straight road at a speed of 15 ft/s. When he passes under the balloon, it is 45 ft above him. How fast is the distance between the boy and the balloon increasing 3 s later?

100. A waterskier skis over the ramp shown in the figure at a speed of 30 ft/s. How fast is she rising as she leaves the ramp?



101. The angle of elevation of the sun is decreasing at a rate of 0.25 rad/h . How fast is the shadow cast by a 400-ft-tall building increasing when the angle of elevation of the sun is $\pi/6$?
102. (a) Find the linear approximation to $f(x) = \sqrt{25 - x^2}$ near 3.
 (b) Illustrate part (a) by graphing f and the linear approximation.
 (c) For what values of x is the linear approximation accurate to within 0.1?
103. (a) Find the linearization of $f(x) = \sqrt[3]{1 + 3x}$ at $a = 0$. State the corresponding linear approximation and use it to give an approximate value for $\sqrt[3]{1.03}$.
 (b) Determine the values of x for which the linear approximation given in part (a) is accurate to within 0.1.

104. Evaluate dy if $y = x^3 - 2x^2 + 1$, $x = 2$, and $dx = 0.2$.

105. A window has the shape of a square surmounted by a semi-circle. The base of the window is measured as having width 60 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum error possible in computing the area of the window.

106–108 Express the limit as a derivative and evaluate.

106. $\lim_{x \rightarrow 1} \frac{x^{17} - 1}{x - 1}$

107. $\lim_{h \rightarrow 0} \frac{\sqrt[4]{16 + h} - 2}{h}$

108. $\lim_{\theta \rightarrow \pi/3} \frac{\cos \theta - 0.5}{\theta - \pi/3}$

109. Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 + \sin x}}{x^3}$.

110. Suppose f is a differentiable function such that $f(g(x)) = x$ and $f'(x) = 1 + [f(x)]^2$. Show that $g'(x) = 1/(1 + x^2)$.

111. Find $f'(x)$ if it is known that

$$\frac{d}{dx}[f(2x)] = x^2$$

112. Show that the length of the portion of any tangent line to the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ cut off by the coordinate axes is constant.

Problems Plus

Before you look at the examples, cover up the solutions and try them yourself first.

EXAMPLE 1 How many lines are tangent to both of the parabolas $y = -1 - x^2$ and $y = 1 + x^2$? Find the coordinates of the points at which these tangents touch the parabolas.

SOLUTION To gain insight into this problem, it is essential to draw a diagram. So we sketch the parabolas $y = 1 + x^2$ (which is the standard parabola $y = x^2$ shifted 1 unit upward) and $y = -1 - x^2$ (which is obtained by reflecting the first parabola about the x -axis). If we try to draw a line tangent to both parabolas, we soon discover that there are only two possibilities, as illustrated in Figure 1.

Let P be a point at which one of these tangents touches the upper parabola and let a be its x -coordinate. (The choice of notation for the unknown is important. Of course we could have used b or c or x_0 or x_1 instead of a . However, it's not advisable to use x in place of a because that x could be confused with the variable x in the equation of the parabola.) Then, since P lies on the parabola $y = 1 + x^2$, its y -coordinate must be $1 + a^2$. Because of the symmetry shown in Figure 1, the coordinates of the point Q where the tangent touches the lower parabola must be $(-a, -(1 + a^2))$.

To use the given information that the line is a tangent, we equate the slope of the line PQ to the slope of the tangent line at P . We have

$$m_{PQ} = \frac{1 + a^2 - (-1 - a^2)}{a - (-a)} = \frac{1 + a^2}{a}$$

If $f(x) = 1 + x^2$, then the slope of the tangent line at P is $f'(a) = 2a$. Thus the condition that we need to use is that

$$\frac{1 + a^2}{a} = 2a$$

Solving this equation, we get $1 + a^2 = 2a^2$, so $a^2 = 1$ and $a = \pm 1$. Therefore the points are $(1, 2)$ and $(-1, -2)$. By symmetry, the two remaining points are $(-1, 2)$ and $(1, -2)$.

EXAMPLE 2 For what values of c does the equation $\ln x = cx^2$ have exactly one solution?

SOLUTION One of the most important principles of problem solving is to draw a diagram, even if the problem as stated doesn't explicitly mention a geometric situation. Our present problem can be reformulated geometrically as follows: For what values of c does the curve $y = \ln x$ intersect the curve $y = cx^2$ in exactly one point?

Let's start by graphing $y = \ln x$ and $y = cx^2$ for various values of c . We know that, for $c \neq 0$, $y = cx^2$ is a parabola that opens upward if $c > 0$ and downward if $c < 0$. Figure 2 shows the parabolas $y = cx^2$ for several positive values of c . Most of them don't intersect $y = \ln x$ at all and one intersects twice. We have the feeling that there must be a value of c (somewhere between 0.1 and 0.3) for which the curves intersect exactly once, as in Figure 3.

To find that particular value of c , we let a be the x -coordinate of the single point of intersection. In other words, $\ln a = ca^2$, so a is the unique solution of the given equation. We see from Figure 3 that the curves just touch, so they have a common tangent line when $x = a$. That means the curves $y = \ln x$ and $y = cx^2$ have the same slope when $x = a$. Therefore

$$\frac{1}{a} = 2ca$$

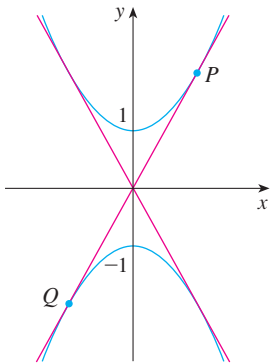


FIGURE 1

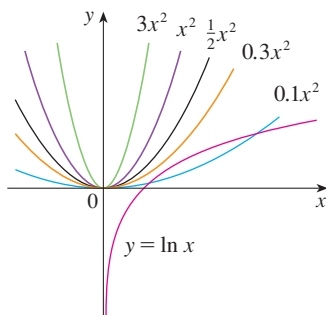


FIGURE 2

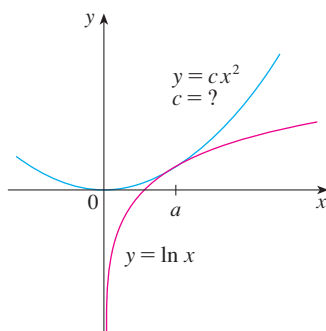


FIGURE 3

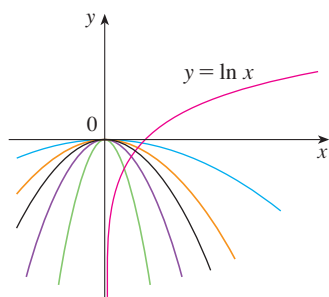


FIGURE 4

Solving the equations $\ln a = ca^2$ and $1/a = 2ca$, we get

$$\ln a = ca^2 = c \cdot \frac{1}{2c} = \frac{1}{2}$$

Thus $a = e^{1/2}$ and

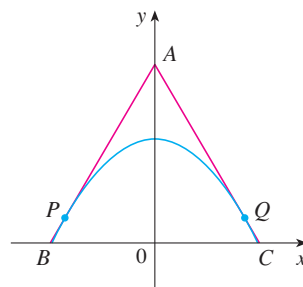
$$c = \frac{\ln a}{a^2} = \frac{\ln e^{1/2}}{e} = \frac{1}{2e}$$

For negative values of c we have the situation illustrated in Figure 4: All parabolas $y = cx^2$ with negative values of c intersect $y = \ln x$ exactly once. And let's not forget about $c = 0$: The curve $y = 0x^2 = 0$ is just the x -axis, which intersects $y = \ln x$ exactly once.

To summarize, the required values of c are $c = 1/(2e)$ and $c \leq 0$.

Problems

- Find points P and Q on the parabola $y = 1 - x^2$ so that the triangle ABC formed by the x -axis and the tangent lines at P and Q is an equilateral triangle. (See the figure.)



- Find the point where the curves $y = x^3 - 3x + 4$ and $y = 3(x^2 - x)$ are tangent to each other, that is, have a common tangent line. Illustrate by sketching both curves and the common tangent.
- Show that the tangent lines to the parabola $y = ax^2 + bx + c$ at any two points with x -coordinates p and q must intersect at a point whose x -coordinate is halfway between p and q .
- Show that

$$\frac{d}{dx} \left(\frac{\sin^2 x}{1 + \cot x} + \frac{\cos^2 x}{1 + \tan x} \right) = -\cos 2x$$

- If $f(x) = \lim_{t \rightarrow x} \frac{\sec t - \sec x}{t - x}$, find the value of $f'(\pi/4)$.
- Find the values of the constants a and b such that

$$\lim_{x \rightarrow 0} \frac{\sqrt[3]{ax + b} - 2}{x} = \frac{5}{12}$$

- Show that $\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x)$.



Graphing calculator or computer required



Computer algebra system required

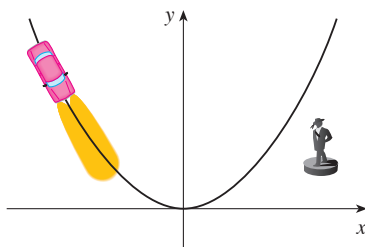


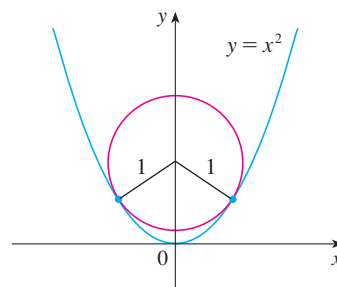
FIGURE FOR PROBLEM 8

8. A car is traveling at night along a highway shaped like a parabola with its vertex at the origin (see the figure). The car starts at a point 100 m west and 100 m north of the origin and travels in an easterly direction. There is a statue located 100 m east and 50 m north of the origin. At what point on the highway will the car's headlights illuminate the statue?

9. Prove that $\frac{d^n}{dx^n} (\sin^4 x + \cos^4 x) = 4^{n-1} \cos(4x + n\pi/2)$.

10. Find the n th derivative of the function $f(x) = x^n/(1-x)$.

11. The figure shows a circle with radius 1 inscribed in the parabola $y = x^2$. Find the center of the circle.



12. If f is differentiable at a , where $a > 0$, evaluate the following limit in terms of $f'(a)$:

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}}$$

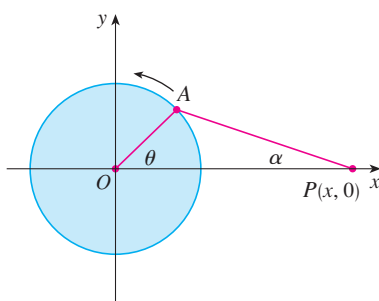


FIGURE FOR PROBLEM 13

13. The figure shows a rotating wheel with radius 40 cm and a connecting rod AP with length 1.2 m. The pin P slides back and forth along the x -axis as the wheel rotates counterclockwise at a rate of 360 revolutions per minute.

- (a) Find the angular velocity of the connecting rod, $d\alpha/dt$, in radians per second, when $\theta = \pi/3$.
 (b) Express the distance $x = |OP|$ in terms of θ .
 (c) Find an expression for the velocity of the pin P in terms of θ .

14. Tangent lines T_1 and T_2 are drawn at two points P_1 and P_2 on the parabola $y = x^2$ and they intersect at a point P . Another tangent line T is drawn at a point between P_1 and P_2 ; it intersects T_1 at Q_1 and T_2 at Q_2 . Show that

$$\frac{|PQ_1|}{|PP_1|} + \frac{|PQ_2|}{|PP_2|} = 1$$

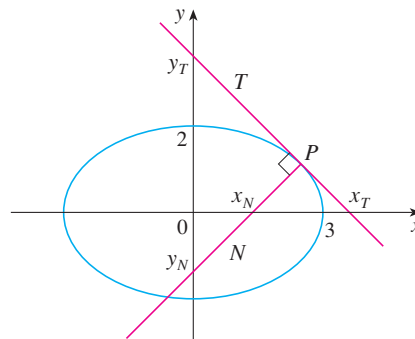
15. Show that

$$\frac{d^n}{dx^n} (e^{ax} \sin bx) = r^n e^{ax} \sin(bx + n\theta)$$

where a and b are positive numbers, $r^2 = a^2 + b^2$, and $\theta = \tan^{-1}(b/a)$.

16. Evaluate $\lim_{x \rightarrow \pi} \frac{e^{\sin x} - 1}{x - \pi}$.

17. Let T and N be the tangent and normal lines to the ellipse $x^2/9 + y^2/4 = 1$ at any point P on the ellipse in the first quadrant. Let x_T and y_T be the x - and y -intercepts of T and x_N and y_N be the intercepts of N . As P moves along the ellipse in the first quadrant (but not on the axes), what values can x_T , y_T , x_N , and y_N take on? First try to guess the answers just by looking at the figure. Then use calculus to solve the problem and see how good your intuition is.



18. Evaluate $\lim_{x \rightarrow 0} \frac{\sin(3+x)^2 - \sin 9}{x}$.
19. (a) Use the identity for $\tan(x - y)$ (see Equation 14b in Appendix D) to show that if two lines L_1 and L_2 intersect at an angle α , then

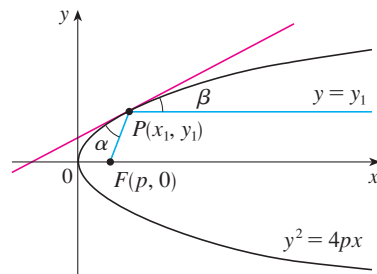
$$\tan \alpha = \frac{m_2 - m_1}{1 + m_1 m_2}$$

where m_1 and m_2 are the slopes of L_1 and L_2 , respectively.

- (b) The **angle between the curves** C_1 and C_2 at a point of intersection P is defined to be the angle between the tangent lines to C_1 and C_2 at P (if these tangent lines exist). Use part (a) to find, correct to the nearest degree, the angle between each pair of curves at each point of intersection.

- (i) $y = x^2$ and $y = (x - 2)^2$
(ii) $x^2 - y^2 = 3$ and $x^2 - 4x + y^2 + 3 = 0$

20. Let $P(x_1, y_1)$ be a point on the parabola $y^2 = 4px$ with focus $F(p, 0)$. Let α be the angle between the parabola and the line segment FP , and let β be the angle between the horizontal line $y = y_1$ and the parabola as in the figure. Prove that $\alpha = \beta$. (Thus, by a principle of geometrical optics, light from a source placed at F will be reflected along a line parallel to the x -axis. This explains why *paraboloids*, the surfaces obtained by rotating parabolas about their axes, are used as the shape of some automobile headlights and mirrors for telescopes.)



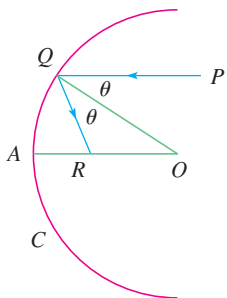


FIGURE FOR PROBLEM 21

21. Suppose that we replace the parabolic mirror of Problem 20 by a spherical mirror. Although the mirror has no focus, we can show the existence of an *approximate* focus. In the figure, C is a semicircle with center O . A ray of light coming in toward the mirror parallel to the axis along the line PQ will be reflected to the point R on the axis so that $\angle PQO = \angle OQR$ (the angle of incidence is equal to the angle of reflection). What happens to the point R as P is taken closer and closer to the axis?

22. If f and g are differentiable functions with $f(0) = g(0) = 0$ and $g'(0) \neq 0$, show that

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}$$

23. Evaluate $\lim_{x \rightarrow 0} \frac{\sin(a + 2x) - 2 \sin(a + x) + \sin a}{x^2}$.

- CAS** 24. (a) The cubic function $f(x) = x(x - 2)(x - 6)$ has three distinct zeros: 0, 2, and 6. Graph f and its tangent lines at the *average* of each pair of zeros. What do you notice?
 (b) Suppose the cubic function $f(x) = (x - a)(x - b)(x - c)$ has three distinct zeros: a , b , and c . Prove, with the help of a computer algebra system, that a tangent line drawn at the average of the zeros a and b intersects the graph of f at the third zero.

25. For what value of k does the equation $e^{2x} = k\sqrt{x}$ have exactly one solution?

26. For which positive numbers a is it true that $a^x \geq 1 + x$ for all x ?

27. If

$$y = \frac{x}{\sqrt{a^2 - 1}} - \frac{2}{\sqrt{a^2 - 1}} \arctan \frac{\sin x}{a + \sqrt{a^2 - 1} + \cos x}$$

$$\text{show that } y' = \frac{1}{a + \cos x}.$$

28. Given an ellipse $x^2/a^2 + y^2/b^2 = 1$, where $a \neq b$, find the equation of the set of all points from which there are two tangents to the curve whose slopes are (a) reciprocals and (b) negative reciprocals.
29. Find the two points on the curve $y = x^4 - 2x^2 - x$ that have a common tangent line.
30. Suppose that three points on the parabola $y = x^2$ have the property that their normal lines intersect at a common point. Show that the sum of their x -coordinates is 0.
31. A *lattice point* in the plane is a point with integer coordinates. Suppose that circles with radius r are drawn using all lattice points as centers. Find the smallest value of r such that any line with slope $\frac{2}{3}$ intersects some of these circles.
32. A cone of radius r centimeters and height h centimeters is lowered point first at a rate of 1 cm/s into a tall cylinder of radius R centimeters that is partially filled with water. How fast is the water level rising at the instant the cone is completely submerged?
33. A container in the shape of an inverted cone has height 16 cm and radius 5 cm at the top. It is partially filled with a liquid that oozes through the sides at a rate proportional to the area of the container that is in contact with the liquid. (The surface area of a cone is $\pi r l$, where r is the radius and l is the slant height.) If we pour the liquid into the container at a rate of 2 cm³/min, then the height of the liquid decreases at a rate of 0.3 cm/min when the height is 10 cm. If our goal is to keep the liquid at a constant height of 10 cm, at what rate should we pour the liquid into the container?

This is a preview.
For the entire book, contact jacob.wu@email.com